Karshi Multidisciplinary International Scientific Journal

Vol. 1(1) (2024), 17–26

ON THE SOLUTION OF THE ILL-POSED CAUCHY PROBLEM FOR ELLIPTIC SYSTEMS OF THE FIRST ORDER

D.A. Juraev, A.A. Tagiyeva, J.D. Bulnes and G.X.-G. Yue

Communicated by Vagif Ibrahimov

Abstract: In this paper, we consider the problem of recovering solutions of matrix factorizations of the Helmholtz equation in a four-dimensional bounded domain from their values on a part of the boundary of this domain, i.e., the Cauchy problem. Based on the Carleman function, an explicit solution of the Cauchy problem for matrix factorizations of the Helmholtz equation is constructed.

Keywords and phrases: The Cauchy problem, regularization, factorization, regular solution, fundamental solution.

MSC 2010 Classifications: Primary 35J46; Secondary 35J56.

1 Introduction

The most actively developing modern area of scientific knowledge is the theory of correctly and incorrectly posed problems, most of which have practical value and require decision making in uncertain or contradictory conditions. The development and justification of methods for solving such a complex class of problems as ill-posed ones is an urgent problem of the present time. The theory of ill-posed problems is an apparatus of scientific research for many scientific areas, such as differentiation of approximately given functions, solving inverse boundary value problems, solving problems of linear programming and control systems, solving degenerate or ill-conditioned systems of linear equations, etc. The concept of a well-posed problem is due to J. Hadamard (1923), who took the point of view that every mathematical problem corresponding to some physical or technological problem must be well-posed. In fact, what physical interpretation can a solution have if an arbitrary small change in the data can lead to large changes in the solution? Moreover, it would be difficult to apply approximation methods to such problems. This put the expediency of studying ill-posed problems in doubt [34]. For ill-posed problems of the question arises: What is meant by an approximate solution? Clearly, it should be so defined that it is stable under small changes of the original information. A second question is: What algorithms are there for the construction of such solutions? Answers to these basic questions were given by A.N. Tikhonov (see [2]).

The concept of conditional correctness first appeared in the work of Tikhonov [2], and then in the studies of Lavrent'ev [37, 38]. In a theoretical study of the conditional correctness (correctness according to Tikhonov) of an ill-posed problem of the existence of a solution and its belonging to the correctness set, it is postulated in the very formulation of the problem. The study of uniqueness issues in a conditionally well-posed formulation does not essentially differ from the study in a classically well-posed formulation, and the stability of the solution from the data of the problem is required only from those variations of the data that do not deduce solutions from the well-posedness set. After establishing the uniqueness and stability theorems in the study of the conditional correctness of ill-posed problems, the question arises of constructing effective solution methods, i.e. construction of regularizing operators. It is known that the Cauchy problem for elliptic equations and for systems of elliptic equations belongs to the class of ill-posed problems (see, for example, [1, 2, 29, 34], [36]-[38], [46]-[48]). Boundary value problems, as well as numerical solutions of some problems, are considered in [3]-[6], [30]-[33], [35, 39, 44], [49]-[51].

Based on the results of previous works [37, 38], [40]-[42], [46, 47] we have constructed the Carleman matrix and based on it the approximate solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. In this article, we find an explicit formula for an approximate solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain of an odd-dimensional space $\mathbb{R}^m$. The case of an even-dimensional space will be considered in other scientific studies of the authors. Our approximate solution formula also includes the construction of a family of fundamental solutions of the Helmholtz operator in space. This family is parametrized by some entire function $K(z)$, the choice of which depends on the dimension of the space. In this work, relying on the results of previous works [7]-[16], we similarly obtain better results with approximate estimates due to a special selection of the function $K(z)$. This helped to get good results when finding an approximate solution based on the Carleman matrix. The Carleman matrix or Carleman function for some elliptic equations and systems was considered in the following studies [7]-[16], [28, 37, 38], [40]-[43], [46, 47].

When solving correct problems, sometimes it is not possible to find the value of the vector function on the entire boundary. Finding the value of a vector function on the entire boundary for systems of elliptic type with constant coefficients (see, for example, [7]-[27]) is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in [37, 38, 46, 47] and subsequently developed in [7]-[16], [28], [40]-[43].

2 Basic information and statement of the Cauchy problem

Let $\mathbb{R}^m$, $(m = 2k+1,\ k \geq 1)$ be the $m-$ dimensional real Euclidean space,

$$\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{R}^m, \quad \eta = (\eta_1, \dots, \eta_m) \in \mathbb{R}^m,$$

$$\zeta' = (\zeta_1, \dots, \zeta_{m-1}) \in \mathbb{R}^{m-1}, \quad \eta' = (\eta_1, \dots, \eta_{m-1}) \in \mathbb{R}^{m-1}.$$

Next, we use the following notation:

$$r = |\eta - \zeta|, \quad \alpha = |\eta' - \zeta'|, \quad z = i\sqrt{a^2 + \alpha^2} + \eta_m, \quad a \geq 0,$$

$$\partial_\zeta = (\partial_{\zeta_1}, \dots, \partial_{\zeta_m})^T, \quad \partial_\zeta = \chi^T, \quad \chi^T = \begin{pmatrix} \chi_1 \\ \dots \\ \chi_m \end{pmatrix} - \text{transposed vector} \quad \chi,$$

$$W(\zeta) = (W_1(\zeta), \dots, W_n(\zeta))^T, \quad v^0 = (1, \dots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m \geq 3,$$

$$E(w) = \begin{Vmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & w_n \end{Vmatrix} - \text{diagonal matrix}, w = (w_1, \dots, w_n) \in \mathbb{R}^n.$$

We also consider a bounded simply-connected domain $\Omega \subset \mathbb{R}^m$, having a piecewise smooth boundary $\partial\Omega = \Sigma \bigcup D$, where Σ is a smooth surface lying in the half-space $\eta_m > 0$ and D is the plane $\eta_m = 0$.

$P(\chi^T)$ is an $(n \times n)-$dimensional matrix satisfying:

$$P^*(\chi^T)P(\chi^T) = E((|\chi|^2 + \lambda^2)v^0),$$

where $P^*(\chi^T)$ is the Hermitian conjugate matrix of $P(\chi^T)$, $\lambda \in \mathbb{R}$, the elements of the matrix $P(\chi^T)$ consist of a set of linear functions with constant coefficients from the complex plane $\mathbb{C}$.

Let us consider the following first order systems of linear partial differential equations with constant coefficients

$$P(\partial_\zeta) W(\zeta) = 0, \tag{2.1}$$

in the domain Ω, where $P(\partial_\zeta)$ is the matrix differential operator of the first-order.

Also consider the set

$$S(\Omega) = \left\{W : \overline{\Omega} \longrightarrow \mathbb{R}^n\right\},$$

here W is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$ and W satisfies the system (2.1).

The Cauchy problem for system (2.1) is formulated as follows:

Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a continuous given function on Σ.

Problem. Suppose $W(\eta) \in S(\Omega)$ and

$$W(\eta)|_{\Sigma} = f(\eta), \quad \eta \in \Sigma. \tag{2.2}$$

Our purpose is to determine the function $W(\eta)$ in the domain Ω when its values are known on Σ.

If $W(\eta) \in S(\Omega)$, then the following integral representation holds

$$W(\zeta) = \int_{\partial\Omega} L(\eta,\zeta;\lambda)W(\eta)ds_\eta, \quad \zeta \in \Omega, \tag{2.3}$$

where

$$L(\eta,\zeta;\lambda) = \left(E\left(\Gamma_m(\lambda r)v^0\right)P^*(\partial_\zeta)\right)P(t^T).$$

Here $t = (t_1,\ldots,t_m)$–is the unit exterior normal, drawn at a point η, the surface $\partial\Omega$, $\Gamma_m(\lambda r)$– is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, $(m = 2k+1,\ k \geq 1)$, where $\Gamma_m(\lambda r)$ defined by the following formula (see [45]):

$$\Gamma_m(\lambda r) = B_m \lambda^{(m-2)/2} \frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}},$$
$$B_m = \frac{1}{2i(2\pi)^{(m-2)/2}}, \quad m = 2k+1, \quad k \geq 1. \tag{2.4}$$

Let $K(z)$ be an entire function taking real values for real z, $(z = a+ib;\ a,b \in \mathbb{R})$ such that

$$K(a) \neq 0, \quad \sup_{b\geq 1}\left|b^p K^{(p)}(z)\right| = N(a,p) < \infty,$$
$$-\infty < a < \infty, \quad p = 0,1,\ldots,m. \tag{2.5}$$

We define the function $\Psi(\eta,\zeta;\lambda)$ at $\eta \neq \zeta$ by the following equality

$$\Psi(\eta,\zeta;\lambda) = \frac{1}{c_m K(\zeta_m)}\frac{\partial^{k-1}}{\partial s^{k-1}}\int_0^\infty \operatorname{Im}\left[\frac{K(z)}{z-\zeta_m}\right]\frac{\cos(\lambda a)}{\sqrt{a^2+\alpha^2}}da,$$
$$m = 2k+1, \quad k \geq 1, \tag{2.6}$$

where $c_m = (-1)^k 2^{-k}(2k-1)!(m-2)\pi\omega_m$, $k \geq 1$; ω_m is the area of a unit sphere in $\mathbb{R}^m$.

In equality (2.6), choosing the function $K(z)$ in the form

$$K(z) = \exp(\sigma z^2), \quad K(\zeta_m) = \exp(\sigma\zeta_m^2), \quad \sigma > 0, \tag{2.7}$$

we get

$$\Psi_\sigma(\eta,\zeta;\lambda) = \frac{e^{-\sigma\zeta_m^2}}{c_m}\frac{\partial^{k-1}}{\partial s^{k-1}}\int_0^\infty \operatorname{Im}\left[\frac{\exp(\sigma z^2)}{z-\zeta_m}\right]\frac{\cos(\lambda a)}{\sqrt{a^2+\alpha^2}}da. \tag{2.8}$$

The integral representation (2.3) will be true if instead $\Gamma_m(\lambda r)$ we substitute the function

$$\Psi_\sigma(\eta,\zeta;\lambda) = \Gamma_m(\lambda r) + G_\sigma(\eta,\zeta;\lambda), \tag{2.9}$$

where $G_\sigma(\eta, \zeta; \lambda)-$ is the regular solution of the Helmholtz equation with respect to the variable η, including the point $\eta = \zeta$.

Then the integral representation (2.3) can be written as:

$$W(\zeta) = \int\limits_{\partial\Omega} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega, \tag{2.10}$$

where

$$L_\sigma(\eta, \zeta; \lambda) = \left(E\left(\Psi_\sigma(\eta, \zeta; \lambda) v^0\right) P^*\left(\partial_\zeta\right)\right) P(t^T).$$

3 Regularized solution of problem (2.1) – (2.2)

Theorem 3.1. *Suppose $W(\eta) \in S(\Omega)$ satisfies the following boundary condition*

$$|W(\eta)| \leq M, \quad \eta \in D. \tag{3.1}$$

If

$$W_\sigma(\zeta) = \int\limits_{\Sigma} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega, \tag{3.2}$$

then the following estimate is true

$$|W(\zeta) - W_\sigma(\zeta)| \leq MC(\zeta)\sigma^{k+1} e^{-\sigma\zeta_m^2}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{3.3}$$

Here and next the functions bounded on compact subsets of the domain Ω, we denote by $C(\zeta)$.

Proof. Here, using the integral representation (2.10) and the equality (3.2), we get the following equality

$$W(\zeta) = \int\limits_{\Sigma} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta + \int\limits_{D} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta =$$

$$= W_\sigma(\zeta) + \int\limits_{D} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega.$$

Using boundary condition (3.1), we estimate the following

$$|W(\zeta) - W_\sigma(\zeta)| \leq \left| \int\limits_{D} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta \right| \leq$$
$$\leq \int\limits_{D} |L_\sigma(\eta, \zeta; \lambda)| \, |W(\eta)| \, ds_\eta \leq M \int\limits_{D} |L_\sigma(\eta, \zeta; \lambda)| \, ds_\eta, \quad \zeta \in \Omega. \tag{3.4}$$

Next, we estimate the integrals $\int\limits_{D} |\Psi_\sigma(\eta, \zeta; \lambda)| \, ds_\eta$, $\int\limits_{D} \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \eta_j} \right| ds_\eta$, $j = \overline{1, m-1}$ and $\int\limits_{D} \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \eta_m} \right| ds_\eta$ on the part D of the plane $\eta_m = 0$.

To do this, first, we separate the imaginary part of the equality (2.8), and as a result we get the following equality

$$\Psi_\sigma(\eta, \zeta; \lambda) = \frac{e^{\sigma(\eta_m^2 - \zeta_m^2)}}{c_m} \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty -\frac{e^{-\sigma(a^2+\alpha^2)} \cos \sigma\sqrt{a^2+\alpha^2}}{a^2+r^2} \cos(\lambda a) \, da + \right.$$
$$\left. + \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \frac{e^{-\sigma(a^2+\alpha^2)} (\eta_m - \zeta_m) \sin \sigma\sqrt{a^2+\alpha^2}}{a^2+r^2} \frac{\cos(\lambda a)}{\sqrt{a^2+\alpha^2}} \, da \right], \quad \zeta_m > 0. \tag{3.5}$$

Taking into account equality (3.5), we have

$$\int_D |\Psi_\sigma(\eta,\zeta;\lambda)|\, ds_\eta \le C(\zeta)\sigma^{k+1}e^{-\sigma\zeta_m^2}, \quad \sigma>1, \quad \zeta\in\Omega, \tag{3.6}$$

Next, we use the following equality to estimate the second integral

$$\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j} = \frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial\eta_j} = 2(y_\eta-\zeta_j)\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}, \tag{3.7}$$
$$s=\alpha^2, \quad j=1,2,\dots,m-1.$$

Given equality (3.5) and equality (3.7), we obtain

$$\int_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j}\right| ds_\eta \le C(\zeta)\sigma^{k+1}e^{-\sigma\zeta_m^2}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{3.8}$$

Now, we estimate the integral $\int_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_m}\right| ds_\eta$.

Taking into account equality (3.5), we obtain

$$\int_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_m}\right| ds_\eta \le C(\zeta)\sigma^{k+1}e^{-\sigma\zeta_m^2}, \quad \sigma>1, \quad \zeta\in\Omega, \tag{3.9}$$

From inequalities (3.6), (3.8), (3.9) and (3.4), we obtain an estimate (3.3). □

Corollary 3.2. *The limiting equality*

$$\lim_{\sigma\to\infty} W_\sigma(\zeta) = W(\zeta),$$

holds uniformly on each compact set from the domain Ω.

4 Estimation of the stability of the solution to the Cauchy problem

Theorem 4.1. *Let* $W(\eta)\in S(\Omega)$ *satisfy condition* (3.1)*, and on a smooth surface* Σ *the inequality*

$$|W(\eta)|\le\delta, \quad 0<\delta<M, \tag{4.1}$$

Then the following estimate is true

$$|W(\zeta)|\le C(\zeta)\sigma^{k+1}M^{1-\frac{\zeta_m^2}{\bar\eta_m^2}}\delta^{\frac{\zeta_m^2}{\bar\eta_m^2}}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{4.2}$$

Proof. Using the integral formula (2.10), we have

$$W(\zeta)=\int_\Sigma L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta + \int_D L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta, \quad \zeta\in\Omega.$$

Then we estimate the following

$$|W(\zeta)|\le\left|\int_\Sigma L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right| + \left|\int_D L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right|, \quad \zeta\in\Omega. \tag{4.3}$$

Given inequality (4.1), we estimate the first integral of inequality (4.3).

$$\left|\int_\Sigma L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right| \le \int_\Sigma |L_\sigma(\eta,\zeta;\lambda)|\,|W(\eta)|\,ds_\eta \le \tag{4.4}$$
$$\le\delta\int_\Sigma |L_\sigma(\eta,\zeta;\lambda)|\,ds_\eta, \quad \zeta\in\Omega.$$

To do this, we estimate the integrals $\int\limits_{\Sigma} |\Psi_\sigma(\eta,\zeta;\lambda)|\, ds_\eta$, $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j}\right| ds_\eta$, $j=\overline{1,m-1}$ and $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_m}\right| ds_\eta$ on a smooth surface Σ.

Taking into account equality (3.5), we have

$$\int\limits_{\Sigma} |\Psi_\sigma(\eta,\zeta;\lambda)|\, ds_\eta \le C(\zeta)\sigma^{k+1} e^{\sigma(\eta_m^2-\zeta_m^2)}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{4.5}$$

To estimate the second integral, on using the equalities (3.5) and (3.7), we obtain

$$\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j}\right| ds_\eta \le C(\zeta)\sigma^{k+1} e^{\sigma(\eta_m^2-\zeta_m^2)}, \quad \sigma>1, \quad \zeta\in\Omega, \tag{4.6}$$

To estimate the integral $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_m}\right| ds_\eta$, on using the equality (3.5), we obtain

$$\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_m}\right| ds_\eta \le C(\zeta)\sigma^{k+1} e^{\sigma(\eta_m^2-\zeta_m^2)}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{4.7}$$

From (4.5) – (4.7) and (4.4), we obtain

$$\left|\int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta\right| \le C(\zeta)\sigma^{k+1}\delta\, e^{\sigma(\eta_m^2-\zeta_m^2)}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{4.8}$$

The following is known

$$\left|\int\limits_{T} N_\sigma(y,x;\lambda) U(y) ds_y\right| \le MC(x)\sigma^{k+1} e^{-\sigma x_m^2}, \quad \sigma>1, \quad x\in G. \tag{4.9}$$

Now taking into account (4.8) – (4.9) and (4.3), we have

$$|W(\zeta)| \le \frac{C(\zeta)\sigma^{k+1}}{2}(\delta\, e^{\sigma\bar{\eta}_m^2}+M)e^{-\sigma\zeta_m^2}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{4.10}$$

Choosing σ from the equality

$$\sigma = \frac{1}{\bar{\eta}_m^2}\ln\frac{M}{\delta}, \tag{4.11}$$

we obtain an estimate (4.2). □

Let $W(\eta)\in S(\Omega)$ and instead $W(\eta)$ on Σ with its approximation $f_\delta(\eta)$ are given, respectively, with an error $0<\delta<M$,

$$\max_{\Sigma} |W(\eta)-f_\delta(\eta)| \le \delta. \tag{4.12}$$

We put

$$W_{\sigma(\delta)}(\zeta) = \int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda) f_\delta(\eta) ds_\eta, \quad \zeta\in\Omega. \tag{4.13}$$

The following is true

Theorem 4.2. *Let $W(\eta)\in S(\Omega)$ on the part of the plane $\eta_m=0$ satisfy condition* (3.1). *Then the following estimate is true*

$$\left|W(\zeta)-W_{\sigma(\delta)}(\zeta)\right| \le C(\zeta)\sigma^{k+1} M^{1-\frac{\zeta_m^2}{\bar{\eta}_m^2}} \delta^{\frac{\zeta_m^2}{\bar{\eta}_m^2}}, \quad \zeta\in\Omega. \tag{4.14}$$

Proof. From the integral formulas (2.10) and (4.13), we have

$$W(\zeta) - W_{\sigma(\delta)}(\zeta) = \int\limits_{\partial\Omega} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta - \int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)f_\delta(\eta)ds_\eta =$$

$$= \int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta + \int\limits_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta - \int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)f_\delta(\eta)ds_\eta =$$

$$= \int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)\left\{W(\eta) - f_\delta(\eta)\right\} ds_\eta + \int\limits_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta.$$

Using conditions (3.1) and (4.12), we estimate the following:

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| = \left|\int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)\left\{W(\eta) - f_\delta(\eta)\right\} ds_\eta\right| +$$

$$+\left|\int\limits_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right| \le \int\limits_{\Sigma} \left|L_\sigma(\eta,\zeta;\lambda)\right| \left|\left\{W(\eta) - f_\delta(\eta)\right\}\right| ds_y +$$

$$+\int\limits_{D} \left|L_\sigma(\eta,\zeta;\lambda)\right| \left|W(\eta)\right| ds_\eta \le \delta \int\limits_{\Sigma} \left|L_\sigma(\eta,\zeta;\lambda)\right| ds_\eta + M \int\limits_{D} \left|L_\sigma(\eta,\zeta;\lambda)\right| ds_\eta.$$

Now, repeating the proof of Theorems 3.1 and 4.1, we obtain

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| \le \frac{C(\lambda,\zeta)\sigma^{k+1}}{2}(\delta\, e^{\sigma\bar{\eta}_m^2} + M)e^{-\sigma\zeta_m^2}.$$

Next, choosing σ from equality (4.11), we obtain an estimate (4.14). □

Corollary 4.3. *The limit equality*

$$\lim_{\delta\to 0} W_{\sigma(\delta)}(\zeta) = W(\zeta),$$

holds uniformly on every compact set from the domain G.

5 Conclusion

In this paper, we have explicitly found a regularized solution to the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain. It is assumed that a solution to the problem exists and is continuously differentiable in a closed domain with exactly given Cauchy data. For this case, an explicit formula for the continuation of the solution is established, as well as a regularization formula for the case when, under the indicated conditions, instead of the Cauchy data, their continuous approximations with a given error in the uniform metric are given. We have obtained a stability estimate for the solution of the Cauchy problem in the classical sense.

Thus, functional $W_{\sigma(\delta)}(\zeta)$ determines the regularization of the solution for problems (2.1) and (2.2).

Acknowledgment

The author expresses his sincere gratitude to the reviewers and editors for their useful comments.

References

[1] A. Bers, F. John, M. Shekhter, *Partial Differential Equations*, Mir, Moscow, (1966).

[2] A.N. Tikhonov, On the solution of ill-posed problems and the method of regularization, *Dokl. Akad. Nauk SSSR*, **151:3**, 501–504 (1963).

[3] A. Shokri, H. Saadat, P-stability, TF and VSDPL technique in Obrechkoff methods for the numerical solution of the Schrödinger equation, *Bull. Iran. Math. Soc.*, **42:3**, 687–706, (2016).

[4] A. Shokri, M. Tahmourasi, A new two-step Obrechkoff method with vanished phase-lag and some of its derivatives for the numerical solution of radial Schrödinger equation and related IVPs with oscillating solutions, *Iranian J. Math. Chem.*, **8:2**, 137–159, (2017).

[5] A. Shokri, M.M. Khalsaraei, S. Noeiaghdam, D.A. Juraev, A new divided difference interpolation method for two-variable functions, *Global and Stochastic Analysis*, **9:23**, 19–26, (2022).

[6] B.C. Corcino, R.B. Corcino, B.A.A. Damgo, J.A.A. Cañete, Integral representation and explicit formula at rational arguments for Apostol - Tangent polynomials, *Symmetry*, **14:1**, 1–10, (2022).

[7] D.A. Juraev, The Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain, *Siberian Electronic Mathematical Reports*, **14**, 752–764, (2017).

[8] D.A. Juraev, On the Cauchy problem for matrix factorizations of the Helmholtz equation in a bounded domain, *Siberian Electronic Mathematical Reports*, **15**, 11–20, (2018).

[9] D.A. Zhuraev, Cauchy problem for matrix factorizations of the Helmholtz equation, *Ukrainian Mathematical Journal*, **69:10**, 1583–1592, (2018).

[10] D.A. Juraev, On the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain in $\mathbb{R}^2$, *Siberian Electronic Mathematical Reports*, **15**, 1865–1877, (2018).

[11] D.A. Juraev, S. Noeiaghdam, Regularization of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation on the plane, *Axioms*, **10:2**, 1–14, (2021).

[12] D.A. Juraev, Solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation on the plane, *Global and Stochastic Analysis*, **8:3**, 1–17, (2021).

[13] D.A. Juraev, S. Noeiaghdam, Modern problems of mathematical physics and their applications, *Axioms*, **11:2**, 1–6, (2022).

[14] D.A. Juraev, Y.S. Gasimov, On the regularization Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain, *Azerbaijan Journal of Mathematics*, **12:1**, 142–161, (2022).

[15] D.A. Juraev, S. Noeiaghdam, Modern problems of mathematical physics and their applications, MDPI, Axioms, Basel, Switzerland, (2022).

[16] D.A. Juraev, On the solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional spatial domain, *Global and Stochastic Analysis*, **9:2**, 1–17, (2022).

[17] D.A. Juraev, The solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain, *Palestine Journal of Mathematics*, **11:3**, 604–613, (2022).

[18] D.A. Juraev, A. Shokri, D. Marian, Solution of the ill-posed Cauchy problem for systems of elliptic type of the first order, *Fractal and Fractional*, **6:7**, 1–11, (2022).

[19] D.A. Juraev, A. Shokri, D. Marian, On an approximate solution of the Cauchy problem for systems of equations of elliptic type of the first order, *Entropy*, **24:7**, 1–18, (2022).

[20] D.A. Juraev, A. Shokri, D. Marian, On the approximate solution of the Cauchy problem in a multidimensional unbounded domain, *Fractal and Fractional*, **6:7**, 1–14, (2022).

[21] D.A. Juraev, A. Shokri, D. Marian, Regularized solution of the Cauchy problem in an unbounded domain, *Symmetry*, **14:8**, 1–16, (2022).

[22] D.A. Juraev, M.M. Cavalcanti, Cauchy problem for matrix factorizations of the Helmholtz equation in the space $\mathbb{R}^m$, *Boletim da Sociedade Paranaense de Matematica*, **41**, 1–12, (2023).

[23] D.A. Juraev, The Cauchy problem for matrix factorization of the Helmholtz equation in a multidimensional unbounded domain, *Boletim da Sociedade Paranaense de Matematica*, **41**, 1–18, (2023).

[24] D.A. Juraev, V. Ibrahimov, P. Agarwal, Regularization of the Cauchy problem for matrix factorizations of the Helmholtz equation on a two-dimensional bounded domain, *Palestine Journal of Mathematics*, **12:1**, 381–403, (2023).

[25] D.A. Juraev, P. Agarwal, A. Shokri, E.E. Elsayed, J.D. Bulnes, On the solution of the ill-posed Cauchy problem for elliptic systems of the first order, *Stochastic Modelling & Computational Sciences*, **3:1**, 1–21, (2023).

[26] D.A. Juraev, S. Noeiaghdam, P. Agarwal, On a regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation, *Turkish World Mathematical Society. Journal of Applied and Engineering Mathematics*, **13:4**, 1311–1326, (2023).

[27] D.A. Juraev, S. Noeiaghdam, P. Agarwal, R.P. Agarwal, On the Cauchy problem for systems of linear equations of elliptic type of the first order in the space $\mathbb{R}^m$, *Turkish World Mathematical Society. Journal of Applied and Engineering Mathematics*, **14:2**, 618–632, (2024).

[28] E.V. Arbuzov, A. L. Bukhgeim, The Carleman formula for the Helmholtz equation on the plane, *Siberian Mathematical Journal*, **47:3**, 425–432, (2006).

[29] G.M. Goluzin, V.M. Krylov, The generalized Carleman formula and its application to the analytic continuation of functions, *Sbornik, Mathematics*, **40:2**, 144–149, (1933).

[30] J. Bulnes, An unusual quantum entanglement consistent with Schrödinger's equation, *Global and Stochastic Analysis*, **9:2**, 79–87, (2022).

[31] J. Bulnes, Solving the heat equation by solving an integro-differential equation, *Global and Stochastic Analysis*, **9:2**, 89–97, (2022).

[32] J.D. Bulnes, D.A. Juraev, J.L. Bonilla, M.A.I. Travassos, Exact decoupling of a coupled system of two stationary Schrödinger equations, *Stochastic Modelling & Computational Sciences*, **3:1**, 23–28, (2023).

[33] J.D. Bulnes, J.L. Bonilla, D.A. Juraev, Klein-Gordon's equation for magnons without non-ideal effect on spatial separation of spin waves, *Stochastic Modelling & Computational Sciences*, **3:1**, 29–37, (2023).

[34] J. Hadamard, *The Cauchy Problem for Linear Partial Differential Equations of Hyperbolic Type*, Nauka, Moscow, (1978).

[35] K. Berdawood, A. Nachaoui, R. Saeed, M. Nachaoui, F. Aboud, An efficient D-N alternating algorithm for solving an inverse problem for Helmholtz equation, *Discrete & Continuous Dynamical Systems–S*, **15:2**, 57–78, (2021).

[36] L.A. Aizenberg, *Carleman's formulas in complex analysis*, Nauka, Novosibirsk, (1990).

[37] M.M. Lavrent'ev, On the Cauchy problem for second-order linear elliptic equations, *Reports of the USSR Academy of Sciences*, **111:2**, 195–197, (1957).

[38] M.M. Lavrent'ev, *On some ill-posed problems of mathematical physics*, Nauka, Novosibirsk, 1962.

[39] N.H. Giang, T.-T. Nguyen, C.C. Tay, L.A. Phuong, T.-T. Dang, Towards predictive Vietnamese human resource migration by machine learning: a case study in northeast Asian countries, *Axioms Anal.*, **11:4**, 1–14, (2022).

[40] N.N. Tarkhanov, Stability of the solutions of elliptic systems, *Funct. Anal. Appl.*, **19:3**, 245–247, (1985).

[41] N.N. Tarkhanov, On the Carleman matrix for elliptic systems, *Reports of the USSR Academy of Sciences*, **284:2**, 294–297, (1985).

[42] N.N. Tarkhanov, A criterion for the solvability of the ill-posed Cauchy problem for elliptic systems, *Dokl. Math.*, **40:2**, 341–345, (1990).

[43] N.N. Tarkhanov, *The Cauchy Problem for Solutions of Elliptic Equations*, Akad. Verl., V. 7, Berlin, (1995).

[44] N. Targyn, D.A. Juraev, Mathematical model of the melting of micro-asperity arising in closed electrical contacts, *Stochastic Modelling & Computational Sciences*, **3:1**, 39–57, (2023).

[45] P.K. Kythe, Fundamental solutions for differential operators and applications, Birkhauser, Boston, (1996).

[46] Sh. Yarmukhamedov, On the Cauchy problem for Laplace's equation, *Dokl. Akad. Nauk SSSR*, **235:2**, 281–283 (1977).

[47] Sh. Yarmukhamedov, The Carleman function and the Cauchy problem for the Laplace equation, *Siberian Mathematical Journal*, **45:3**, 702–719, (2004).

[48] T. Carleman, *Les Fonctions Quasi Analytique*, Gautier-Villars et Cie., Paris, (1926).

[49] V.R. Ibrahimov, G. Mehdiyeva, M.N. Imanova, D.A. Juraev, Application of the bilateral hybrid methods to solving initial - value problems for the Volterra integro-differential equations, *WSEAS Transactions on Mathematics*, **22**, 781–791, (2023).

[50] V.R. Ibrahimov, X.G. Yue, D.A. Juraev, On some advantages of the predictor-corrector methods, *IETI Transactions on Data Analysis and Forecasting (iTDAF)*, **1:4**, 79–893, (2023).

[51] Yu. Fayziyev, Q. Buvaev, D. Juraev, N. Nuralieva, Sh. Sadullaeva, The inverse problem for determining the source function in the equation with the Riemann-Liouville fractional derivative, *Global and Stochastic Analysis*, **9:2**, 43–52, (2022).

Author information

D.A. Juraev, 1. Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, UZBEKISTAN; 2. Department of Mathematics, Anand International College of Engineering, Jaipur, 303012, INDIA.
E-mail: `juraevdavron12@gmail.com`

A.A. Tagiyeva, Presidium of Azerbaijan National Academy of Science, Baku, AZ1001, AZERBAIJAN.
E-mail: aidatagiyeva1984@gmail.com

J.D. Bulnes, Department Ciências Exatas e Tecnologia, Universidade Federal do Amapá, Macapá, 68903-419, BRAZIL; 2. Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, UZBEKISTAN.
E-mail: bulnes@unifap.br

G.X.-G. Yue, Department Computer Science and Engineering, School of Sciences, European University Cyprus, Engomi, 22006, CYPRUS.
E-mail: xgyue@ieti.net

Received: 01.03.2024
Accepted: 01.05.2024
Published: 30.06.2024