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NEW ASPECTS OF CERTAIN SPECIAL FUNCTIONS WITH APPLICATIONS

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Abstract: Throughout the history of natural science, special functions (SFs) have been a powerful instrument in the solution of a wide variety of important problems in various fields such as physics, engineering, biology, medicine, economics, and finance. The reduction of any given applied problem to the evaluation of special functions has always been, and still is, looked on as indicating a strong penetration into the essence of the problem. Almost all the familiar special functions have arisen from a wide diversity of applied problems, so the study of their properties and their applications has engaged not only mathematics but also physicists, astronomers, engineers, and other specialists.

Furthermore, fractional calculus of special functions, which perform fractional differentiation or integration of functions, is gaining popularity due to its numerous scientific, technological, and engineering applications. So, the current review article dives into recent mathematical findings on generalizations of special functions and polynomials related to various fractional calculus operators (FCOs). Also emphasized is their importance and extensive utility in dealing with the most well-known topics: integral transformations, initial value problems, and kinetic equations.

More precisely, we discuss analytic properties and numerical exemplifications of extensions Beta and hypergeometric functions associated with fractional calculus operators. Moreover, some developments and applications of orthogonal matrix polynomials, such as the generalized Bessel matrix polynomials and the generalized Jacobi matrix polynomials, have been considered. Furthermore, novel generalizations of fractional kinetic equations involving certain special functions and their solutions using the various integral transforms have been shown. Finally, some important points that can be suitable to be future works are summarized.

Keywords and phrases: Special functions, fractional operators, integral transforms, Fractional kinetic equations.

MSC 2020 Classifications: 33C20, 33C20, 26A33, 35R11.

1 Introduction

1.1 What are the special functions?

The concept of special functions (SFs) refers to a class of mathematical functions that appear in a variety of situations, most notably in differential equation, mathematical physics, chemistry, and other fields, see, for instance, Sneddon (1956)[1], Nikiforov and Uvarov (1988)[2], and Bell (2004)[3]. These functions frequently possess unique qualities, behaviors, or forms that make them very valuable in applications. Moreover, the term special functions is applied to functions that are not trivial. In some sense the polynomials are the most special of all to the extent of being boring. The next simplest is the family of exponential functions the usual real exponential, the trigonometric and hyperbolic functions, and the general complex exponential that incorporates all of them and their inverse functions. These may have been supposed to be boring but, as we shall see, they are interesting. They can be taken as solutions of first-order differential equations. The next lot of special functions is the solutions of the linear second-order differential equations of mathematical physics and engineering. These include the Legendre polynomials and functions, the Laguerre, the Hermite, the Chebyshev, the Jacobi,the Gegenbauer, the Bessel and related functions and some others. All of them arise as special cases of the generalized hypergeometric functions $\mathcal{U} = {}_p\mathbf{F}_q$ of p numerator and q denominator parameters (see, Dwork

(1990) [4]) that are solutions of the differential equation

$$z\prod_{k=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\alpha_k\right)\mathcal{U}=z\frac{\mathrm{d}}{\mathrm{d}z}\prod_{k=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\beta_k-1\right)\mathcal{U}, \tag{1.1}$$

where

$${}_p\mathbf{F}_q\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_p \\ & & & ;z \\ \beta_1 & \beta_2 & \cdots & \beta_q\end{bmatrix}=\sum_{m=0}^{\infty}\frac{(\alpha_1)_m\cdots(\alpha_p)_m}{(\beta_1)_m\cdots(\beta_q)_m}\frac{z^m}{m!}, \tag{1.2}$$

and $(a)_m$ is the usual Pochhammer symbol (or the rising factorial). For $p=2$ and $q=1$, we get the Gauss hypergeometric function ${}_2\mathbf{F}_1$.

The other functions are those that arose in the theory of numbers. It is of interest to follow them in more detail. The first of them, the Gamma function (*Euler integral of the second kind*), a generalization of the factorial function defined for integers to the real domain and thence to the complex domain (see, Davis (1959) [5]):

$$\Gamma(\varkappa):=\int_0^\infty e^{-y}\,y^{\varkappa-1}\,dy,\quad \mathrm{Re}(\varkappa)>0. \tag{1.3}$$

The function Γ is closely related to the Beta function (*Euler integral of the first kind*)

$$\begin{aligned}\mathrm{B}(\varkappa_1,\varkappa_2)&=\int_0^1 y^{\varkappa_1-1}(1-y)^{\varkappa_2-1}\,dy\\&=\frac{\Gamma(\varkappa_1)\Gamma(\varkappa_2)}{\Gamma(\varkappa_1+\varkappa_2)}\qquad \mathrm{Re}(\varkappa_1,\varkappa_2)>0.\end{aligned} \tag{1.4}$$

By using the Gamma function (1.3). Swedish Mathematician Gosta Mittag-Leffler (1903) [6] introduced the following Mittag-Leffler function of one parameter

$$\mathbf{E}_\alpha(y)=\sum_{r=0}^{\infty}\frac{y^r}{\Gamma(r\alpha+1)},\quad \mathrm{Re}(\alpha)>0. \tag{1.5}$$

This function is the direct generalization of the exponential function and is known as the *queen of the functions of fractional calculus*. For other generalizations, properties, and applications of the Mittag-Leffler functions, the reader might consult the work by Haubold et al. (2001)[7].

Further, SFs are often highly studied, and their properties such as differential (or difference) equations, asymptotic behavior, recurrence relations, integral formulas, orthogonality, Rodrigues' formulas, and generating functions, are extensively documented. These attributes render them effective tools for swiftly addressing intricate issues.

On top of that, recently special matrix functions (SMFs) appear in connection with statistics, Lie group theory, and theoretical physics. Many authors have dealt with orthogonal matrix polynomials and their applications, For examples, Hermite, Laguerre, Chebyshev, Jacobi, Gegenbauer, Bessel and Legendre matrix polynomials were introduced and established, see, tutorial survey by Abdalla (2020) [8].

Next, further details will be provided about the importance of relationships between SFs and both fractional calculus operators (FCOs) and integral transforms (ITs).

1.2 Fractional calculus operators

The notion "Fractional Calculus" (FC) or "Fractional Analysis" is used for the extension of the Calculus (Analysis), when the order of integration and differentiation can be an arbitrary number (fractional, irrational, complex), that is, not obligatory integer. The FC is nowadays one of the most rapidly growing subjects of mathematical analysis in spite of the fact that it is nearly 300 years old. Where the giants of mathematics, W. Leibniz (1697), L. Euler (1730), thought about the possibility to perform differentiation of non-integer order.

The real birth and far-reaching development of the FC is due to numerous attempts of mathematicians during the nineteenth century to beginning of twentieth century. It is practically impossible to name all important contributions made in construction of early stages of building of the FC. For example, see the survey by Machado and Kiryakova (2010)[9, 10], Machado et al. (2010)[11].

In the same vein, fractional calculus operators (FCOs), which handle non-integer (fractional) ordering on functions as a type integral transform convolutions. It is an extension of classical operators, which originated and developed with the pioneering contributions of Lacroix (1819), Liouville (1832), Letnikov (1868), Riemann (1876), Hadamard (1892), Weyl (1919), Caputo (1967) and numerous others [12]. In the literature, FCOs are defined in a wide variety of ways. Two widely used operators are the Riemann-Liouville and Caputo fractional operators [13]. Nevertheless, among the monographs developing the theory of FCOs involving various special functions and presenting some applications, we have to point out works by, Oldham and Spanier (1974) [14], Lovoie (1976)[15] Kilbas (2005) [16], Kiryakova (2008)-(2020)[17, 18], Srivastava (2015)[19], Kochubei (2019)[20], Sandev (2022)[21], and Singh (2023)[22].

Let us now begin with some various representations for FCOs.

The Riemann-Liouville-Caputo fractional operators

The Riemann-Liouville fractional integral operator (RLFIO) of order ν, introduced by Bernhard Riemann and Joseph Liouville, is defined as

$$ {}_0\mathbf{D}_w^{-\nu} f(w) := (\mathbf{I}^\nu f)(w) := \frac{1}{\Gamma(\nu)} \int_0^w (w-u)^{\nu-1} f(u)du, \quad \text{Re}(\nu) > 0, \tag{1.6} $$

where Γ denots the Gamma function defined in (1.3), and the Riemann-Liouville fractional differential operator (RLFDO) of order μ, defined as

$$ \mathbf{D}_w^\mu f(w) := \mathbf{D}^n \Big(\mathbf{I}^{n-\mu} f(w) \Big), \quad n-1 < \text{Re}(\mu) < n. \tag{1.7} $$

In 1967, Italian Caputo defined the Caputo fractional derivative operator (CFDO) as

$$ \mathbf{D}_w^\mu f(w) := \frac{1}{\Gamma(n-\mu)} \int_0^w (w-u)^{n-\mu-1} \frac{d^n}{du^n} f(u)dt, \tag{1.8} $$

where $n-1 < \text{Re}(\mu) < n$.

Recently, many generalizations of RLFOs and CFDOs have been considered, such as the Hilfer, Hadamard, Katugampola, Hilfer-Caputo, and Hadamard-Caputo fractional operators, extend of (1.6), (1.7) and (1.8) by introducing additional parameters or kernel functions. In particular, Mubeen and Habibullah (2012) [23] defined the $k-$RLFIO by

$$ \Big(\mathbf{I}_k^\upsilon f(\tau) \Big)(x) = \frac{1}{k\Gamma^k(\upsilon)} \int_0^x f(\tau)(x-\tau)^{\frac{\upsilon}{k}-1} \, d\tau; \quad \upsilon, k > 0. \tag{1.9} $$

Due to this, the $k-$ RLFIO of order υ introduced by Rahman et al. (2020) [**?**] as

$$ \mathbf{D}_k^\upsilon \Big\{ f(\eta) \Big\} = D \Big(\mathbf{I}_k^{(1-\upsilon)} f(\eta) \Big); \quad 0 < \upsilon \le 1, \quad D = \frac{d}{d\eta}. \tag{1.10} $$

where $\Gamma^k(y)$ is the $k-$Gamma function

$$\Gamma^k(y) = \int_0^\infty u^{y-1} e^{-\frac{y^k}{k}} du, \quad y \in \mathbb{C} \setminus k\mathbb{Z}^-. \tag{1.11}$$

These formulas have been applied in many recent works (see, e.g., [24, 25, 26])

The Hadamard fractional operators

Definition 1.1. [27] Let $0 \leq a \leq b \leq \infty$, be finite or infinite interval of the half-axis $\mathbb{R}^+$. The Hadamard fractional integral operators of order $\alpha \in \mathbb{C}$ are defined by

$$\begin{aligned} \left(\mathcal{HI}_{a+}^{\alpha}\varphi\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1} \varphi(t)\frac{dt}{t}, \quad a < x < b, \\ \left(\mathcal{HI}_{b-}^{\alpha}\varphi\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log\frac{t}{x}\right)^{\alpha-1} \varphi(t)\frac{dt}{t}, \quad a < x < b. \end{aligned} \tag{1.12}$$

The left-sided and right-sided Hadamard fractional derivative operators of order $\alpha \in \mathbb{C}$, $n \in \mathbb{N}$ with $\mathrm{Re}(\alpha) \geq 0$ on (a, b) and $a < x < b$ are defined by

$$\begin{aligned} \left(\mathcal{HD}_{a+}^{\alpha}\varphi\right)(x) &= \delta^n \left(\mathcal{HI}_{a+}^{n-\alpha}\varphi\right)(x) \\ &= \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1} \frac{\varphi(t)dt}{t}, \\ \left(\mathcal{HD}_{b-}^{\alpha}\varphi\right)(x) &= (-\delta)^n \left(\mathcal{HI}_{b-}^{n-\alpha}\varphi\right)(x) \\ &= \left(-x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^b \left(\log\frac{t}{x}\right)^{n-\alpha-1} \frac{\varphi(t)dt}{t}. \end{aligned} \tag{1.13}$$

If $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ and $0 < a < b < \infty$, then we have

$$\begin{aligned} &\left(\mathcal{HI}_{a+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}, \\ &\text{and} \\ &\left(\mathcal{HD}_{a+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}. \end{aligned} \tag{1.14}$$

The Marichev-Saigo-Maeda fractional operators

In 1974, Marichev [28] introduced fractional integral operators as Mellin type convolution operator with the Appell function F_3 in their kernel. In the middle of the 1990s, these fractional integral operators were rediscovered and studied by Saigo (1978-1980) [29, 30], extended and investegated later after that by Saigo and Maeda (1996) [31] and by Saigo and Saxena (2001) [32] as generalizations of the celebrated Saigo fractional integral operators.

The generalized fractional calculus operators (the Marichev-Saigo-Maeda operators) involving the Appell's function or Horn's F_3 function in the kernel are defined as follows:

Definition 1.2. Let $\sigma, \sigma', v, \nu', \eta \in \mathbb{C}$ and $x > 0$, then for $\mathrm{Re}(\eta) > 0$

$$\begin{aligned} &\left(I_{0,x}^{\sigma,\sigma',v,v',\eta} f\right)(x) \\ &= \frac{x^{-\sigma}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\sigma'} F_3\left(\sigma,\sigma',v,v';\eta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)dt, \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} &\left(I_{x,\infty}^{\sigma,\sigma',v,v',\eta} f\right)(x) \\ &= \frac{x^{-\sigma'}}{\Gamma(\eta)} \int_x^\infty (t-x)^{\eta-1} t^{-\sigma} F_3\left(\sigma,\sigma',v,v';\eta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)dt, \end{aligned} \tag{1.16}$$

provided that the function $f(t)$ is so constrained such that the integrals in Equations (1.15) and (1.16) exist.

In Equations (1.15) and (1.16), F_3 denotes the Appell's hypergeometric function [33] in two variables defined as:

$$\begin{aligned} &F_3\left(\sigma,\sigma',v,v';\eta;x,y\right) \\ &= \sum_{m,n=0}^{\infty} \frac{(\sigma)_m\left(\sigma'\right)_n(\nu)_m\left(v'\right)_n}{(\eta)_{m+n}} \frac{x^m}{m!}\frac{x^n}{n!}, \quad (\max\{|x|,|y|\}<1). \end{aligned} \tag{1.17}$$

Then the above fractional integral operators in Equations (1.15) and (1.16), can be written as follows:

$$\begin{gathered} \left(I_{0,x}^{\sigma,\sigma',\nu,v',\eta} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{\sigma,\sigma',\nu+k,v',\eta+k} f\right)(x) \\ (\operatorname{Re}(\eta)<0; k=[-\operatorname{Re}(\eta)+1]), \end{gathered} \tag{1.18}$$

and

$$\begin{gathered} \left(I_{x,\infty}^{\sigma,\sigma',v,v',\eta} f\right)(x) = \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{\sigma,\sigma',v,\nu'+k,\eta+k} f\right)(x) \\ (\operatorname{Re}(\eta)<0; k=[-\operatorname{Re}(\eta)+1]). \end{gathered} \tag{1.19}$$

Remark 1.3. It is worthy to note that the Appell function defined in Equation (1.17) reduces to the Gauss hypergeometric function ${}_2\mathbf{F}_1$ as given in the following relations:

$$F_3(\sigma,\eta-\sigma,v,\eta-v;\eta;x,y) = {}_2\mathbf{F}_1(\sigma,v;\eta;x+y-xy),$$

also we have

$$F_3\left(\sigma,0,v,v',\eta;x,y\right) = {}_2\mathbf{F}_1(\sigma,v;\eta;x),$$

and

$$F_3\left(0,\sigma',v,v',\eta;x,y\right) = {}_2\mathbf{F}_1\left(\sigma',v';\eta;y\right).$$

The corresponding Marichev-Saigo-Maeda fractional differential operators are given as follows:

Definition 1.4. Let $\sigma,\sigma',v,v',\eta \in \mathbb{C}$ and $x>0$, Then

$$\begin{aligned} &\left(D_{0,x}^{\sigma,\sigma',v,v',\eta} f\right)(x) = \left(I_{0,x}^{-\sigma',-\sigma,-v',-v,-\eta} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\sigma',-\sigma,-v'+k,-v,-\eta+k} f\right)(x), \quad (\operatorname{Re}(\eta)>0; k=[\operatorname{Re}(\eta)]+1) \\ &= \frac{1}{\Gamma(k-\eta)}\left(\frac{d}{dx}\right)^k (x)^{\sigma'} \int_0^x (x-t)^{k-\eta-1} t^{\sigma} \\ &\times F_3\left(-\sigma',-\sigma,k-v',-v;k-\eta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)dt, \end{aligned} \tag{1.20}$$

and

$$
\begin{aligned}
&\left(D_{x,\infty}^{\sigma,\sigma',v,v',\eta}f\right)(x)=\left(I_{x,\infty}^{-\sigma',-\sigma,-v',-v,-\eta}f\right)(x)\\
&=\left(-\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{-\sigma',-\sigma,-v',-v+k,-\eta+k}f\right)(x),\quad (\text{Re}(\eta)>0;k=[\text{Re}(\eta)]+1)\\
&=\frac{1}{\Gamma(k-\eta)}\left(-\frac{d}{dx}\right)^{k}(x)^{\sigma}\int_x^{\infty}(t-x)^{k-\eta-1}t^{\sigma'}\\
&\quad\times F_3\left(-\sigma',-\sigma,-v',k-v;k-\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt.
\end{aligned}
\tag{1.21}
$$

In view of the above reduction formula as given in Equation (2.7), the generalized fractional calculus operators reduce to the Saigo operators [**?**] defined as follows:

Definition 1.5. Let $x>0,\sigma,\nu,\eta\in\mathbb{C}$ and $\text{Re}(\sigma)>0$, then

$$\left(I_{0,x}^{\sigma,v,\eta}f\right)(x)=\frac{x^{-\sigma-v}}{\Gamma(\sigma)}\int_0^x(x-t)^{\sigma-1}{}_2F_1\left(\sigma+v,-\eta;\sigma;1-\frac{t}{x}\right)f(t)dt,$$

and

$$\left(I_{x,\infty}^{\sigma,v,\eta}f\right)(x)=\frac{1}{\Gamma(\sigma)}\int_x^{\infty}(t-x)^{\sigma-1}t^{-\sigma-v}{}_2F_1\left(\sigma+v,-\eta;\sigma;1-\frac{x}{t}\right)f(t)dt.$$

The Saigo fractional integral operators, given in Equations (2.11) and (2.12) can also be written as:

Let $x>0,\sigma,v,\eta\in\mathbb{C}$, then

$$
\begin{aligned}
&\left(I_{0,x}^{\sigma,v,\eta}f\right)(x)=\left(\frac{d}{dx}\right)^{k}\left(I_{0,x}^{\sigma+k,v-k,\eta-k}f\right)(x)\\
&(\text{Re}(\sigma)<0;k=[\text{Re}(-\sigma)]+1),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(I_{x,\infty}^{\sigma,v,\eta}f\right)(x)=\left(-\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{\sigma-k,\nu-k,\eta}f\right)(x)\\
&(\text{Re}(\sigma)<0;k=[\text{Re}(-\sigma)]+1).
\end{aligned}
$$

The corresponding Saigo fractional differential operators are defined as:

Definition 1.6. Let $\sigma,v,\eta\in\mathbb{C}$ and $x>0$. Then

$$
\begin{aligned}
\left(D_{0,x}^{\sigma,v,\eta}f\right)(x)=\left(I_{0,x}^{-\sigma,-v,\sigma+\eta}f\right)(x)&=\left(\frac{d}{dx}\right)^{k}\left(I_{0,x}^{-\sigma+k,-v-k,\sigma+\eta-k}f\right)(x)\\
&(\text{Re}(\sigma)>0;k=[\text{Re}(\sigma)]+1),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(D_{x,\infty}^{\sigma,\nu,\eta}f\right)(x)=\left(I_{x,\infty}^{-\sigma,-\nu,\sigma+\eta}f\right)(x)&=\left(-\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{-\sigma+k,-\nu-k,\sigma+\eta}f\right)(x)\\
&(\text{Re}(\sigma)>0;k=[\text{Re}(\sigma)]+1),
\end{aligned}
$$

where $[x]$ denotes the integer part of x.

If we take $v = 0$ in Equations (2.11), (2.12), (2.15) and (2.16), we get the socalled Erdélyi-Kober fractional integral and derivative operators defined as follows[34]:

Definition 1.7. Let $x > 0, \sigma, \eta \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > 0$, then

$$\left(I_{0,x}^{\sigma,\eta} f\right)(x) = \frac{x^{-\sigma-\eta}}{\Gamma(\sigma)} \int_0^x (x-t)^{\sigma-1} t^{\eta} f(t) dt,$$

and

$$\left(I_{x,\infty}^{\sigma,\eta} f\right)(x) = \frac{x^{\eta}}{\Gamma(\sigma)} \int_x^\infty (t-x)^{\sigma-1} t^{-\sigma-\eta} f(t) dt.$$

provided that integrals in Equations (2.17) and (2.18) converge.

The corresponding derivative operators are defined as:

Definition 1.8. Let $x > 0, \sigma, \eta \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > 0$, then

$$\begin{aligned}\left(D_{0,x}^{\sigma,\eta} f\right)(x) =& x^{-\eta} \left(\frac{d}{dx}\right)^k \frac{1}{\Gamma(k-\sigma)} \int_0^x t^{\sigma+\eta} (x-t)^{k-\sigma-1} f(t) dt \\ =& \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\sigma+k,-\sigma,\sigma+\eta-k} f\right)(x), \quad (k = [\mathrm{Re}(\sigma)] + 1),\end{aligned}$$

and

$$\begin{aligned}\left(D_{x,\infty}^{\sigma,\eta} f\right)(x) =& x^{\eta+\sigma} \left(\frac{d}{dx}\right)^k \frac{1}{\Gamma(k-\sigma)} \int_x^\infty t^{-\eta} (t-x)^{k-\sigma-1} f(t) dt \\ =& (-1)^k \left(\frac{d}{dx}\right)^k \left(I_{x,\infty}^{-\sigma+k,-\sigma,\sigma+\eta} f\right)(x), \quad (k = [\mathrm{Re}(\sigma)] + 1).\end{aligned}$$

When $v = -\sigma$, the operators in Equations (2.11), (2.12), (2.15) and (2.16) give the Riemann-Liouville and the Weyl fractional integral operators are defined as follows:

Definition 1.9. Let $x > 0, \sigma \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > 0$, then

$$\left(I_{0,x}^{\sigma} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_0^x (x-t)^{\sigma-1} f(t) dt,$$

and

$$\left(I_{x,\infty}^{\sigma} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_x^\infty (t-x)^{\sigma-1} f(t) dt.$$

provided that both integrals converge.

The corresponding derivative operators are defined as follows:

Definition 1.10. Let $x > 0, \sigma \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > 0$, then

$$\begin{aligned}\left(D_{0,x}^{\sigma} f\right)(x) &= \left(\frac{d}{dx}\right)^k \frac{1}{\Gamma(k-\sigma)} \int_0^x (x-t)^{k-\sigma-1} f(t) dt \\ &= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{k-\sigma} f\right)(x), \quad (k = [\mathrm{Re}(\sigma)] + 1),\end{aligned}$$

and

$$\left(D_{x,\infty}^{\sigma} f\right)(x) = (-1)^k \left(\frac{d}{dx}\right)^k \frac{1}{\Gamma(k-\sigma)} \int_x^{\infty} (t-x)^{k-\sigma-1} f(t)dt$$
$$= (-1)^k \left(\frac{d}{dx}\right)^k \left(I_{x,\infty}^{k-\sigma} f\right)(x), \quad (k = [\mathrm{Re}(\sigma)] + 1).$$

For more detail about such operators along with their properties and applications, one may refer [35, 36, 37].

Power functions formulas of the above discussed fractional operators required for our present study are given in the following lemmas (see [30, 31, 32]):

Lemma 1.11. *Let* $\sigma, \sigma', v, \nu', \eta$ *and* $\rho \in \mathbb{C}, x > 0$ *be such that* $\mathrm{Re}(\eta) > 0$*, then the following formulas hold true:*

$$\left(I_{0,x}^{\sigma,\sigma',v,v',\eta}\, t^{\rho-1}\right)(x)$$
$$= \frac{\Gamma(\rho)\Gamma\left(\rho+\eta-\sigma-\sigma'-v\right)\Gamma\left(\rho+v'-\sigma'\right)}{\Gamma\left(\rho+v'\right)\Gamma\left(\rho+\eta-\sigma-\sigma'\right)\Gamma\left(\rho+\eta-\sigma'-v\right)} x^{\rho+\eta-\sigma-\sigma'-1}$$
$$(\mathrm{Re}(\rho) > \max\left\{0, \mathrm{Re}\left(\sigma+\sigma'+v-\eta\right) \mathrm{Re}\left(\sigma'-v'\right)\right\}),$$

and

$$\left(I_{x,\infty}^{\sigma,\sigma',\nu,v',\eta}\, t^{\rho-1}\right)(x)$$
$$= \frac{\Gamma(1-\rho-v)\Gamma\left(1-\rho-\eta+\sigma+\sigma'\right)\Gamma\left(1-\rho-\eta+\sigma+v'\right)}{\Gamma(1-\rho)\Gamma\left(1-\rho-\eta+\sigma+\sigma'+\nu'\right)\Gamma(1-\rho+\sigma-v)} x^{\rho+\eta-\sigma-\sigma'-1}$$
$$(\mathrm{Re}(\rho) < 1 + \min\left\{\mathrm{Re}(-v), \mathrm{Re}\left(\sigma+\sigma'-\eta\right), \mathrm{Re}\left(\sigma+v'-\eta\right)\right\}).$$

Lemma 1.12. *Let* $\sigma, \sigma', v, \nu', \eta$ *and* $\rho \in \mathbb{C}, x > 0$ *be such that* $\mathrm{Re}(\eta) > 0$*, then the following formulas hold true:*

$$\left(D_{0,x}^{\sigma,\sigma',v,v',\eta}\, t^{\rho-1}\right)(x)$$
$$= \frac{\Gamma(\rho)\Gamma\left(\rho-\eta+\sigma+\sigma'+v'\right)\Gamma(\rho-v+\sigma)}{\Gamma(\rho-v)\Gamma\left(\rho-\eta+\sigma+\sigma'\right)\Gamma\left(\rho-\eta+\sigma+v'\right)} x^{\rho-\eta+\sigma+\sigma'-1}$$
$$(\mathrm{Re}(\rho) > \max\left\{0, \mathrm{Re}\left(\eta-\sigma-\sigma'-v'\right), \mathrm{Re}(\nu-\sigma)\right\}),$$

and

$$\left(D_{x,\infty}^{\sigma,\sigma',v,v',\eta}\, t^{\rho-1}\right)(x)$$
$$= \frac{\Gamma\left(1-\rho+v'\right)\Gamma\left(1-\rho+\eta-\sigma-\sigma'\right)\Gamma\left(1-\rho+\eta-\sigma'-v\right)}{\Gamma(1-\rho)\Gamma\left(1-\rho+\eta-\sigma-\sigma'-v\right)\Gamma\left(1-\rho-\sigma'+v'\right)} x^{\rho-\eta+\sigma+\sigma'-1}$$
$$(\mathrm{Re}(\rho) < 1 + \min\left\{\mathrm{Re}\left(v'\right), \mathrm{Re}\left(\eta-\sigma-\sigma'\right), \mathrm{Re}\left(\eta-\sigma'-v\right)\right\}).$$

2 Integral Transforms

The integral transform (IT) is a mathematical method for translating a differential equation into an algebraic equation. By using this method, a challenging mathematical issue can be reduced to a more manageable one. By integrating the result of a function and another function, known as the kernel of the integral transform, the integral transform mathematical operator can create a result function. The integral transform can be expressed generally as follows:

$$\mathfrak{F}(\varkappa) = \mathbb{T}(f(y)) = \int f(y)\mathbf{K}(y,\varkappa)dy,$$

where, $\mathfrak{F}(\varkappa)$ represents the function produced by the integral transform and $\mathbf{K}(y,\varkappa)$ a kernel function. The inverse integral transform is given by

$$f(y) = \int \mathbb{T}(f(\varkappa))\mathbf{K}^{-1}(\varkappa, y)d\varkappa,$$

where $\mathbf{K}^{-1}(\varkappa, y)$ is the kernel of the inverse integral transform.

We will now provide a discussion of a number of significant integral transforms.

2.1 Fourier transform

The Fourier transform, the Fourier cosine transform, and the Fourier sine transform can be effectively used for solving differential and integral equations. These transforms are also useful in evaluating integrals involving special functions. The reader may refer, for example to, [35, 36, 37, 38].

If f is an absolutely integrable function, the Fourier transforms are defined by [35, 36]

$$F\{f(t)\} = F(k) = \int_{-\infty}^{\infty} f(t)e^{-2\pi kit}dt, \quad k \in \mathbb{R},$$

and

$$F^{-1}\{F(k)\} = f(t) = \int_{-\infty}^{\infty} F(k)e^{2\pi ikt}dk.$$

Further, the fractional Fourier transform (FFT) of order β; $0 < \beta \leq 1$, is defined as

$$\varphi_\beta(\omega) = \mathfrak{F}_\beta[\varphi](\omega) = \int_R e^{i\omega^{\frac{1}{\beta}}\xi}\varphi(\xi)d\xi, \quad i = \sqrt{-1}. \tag{2.1}$$

2.2 Laplace transform

The Laplace transform of $f(t)$ is formally defined by [35, 36]

$$L\{f(t)\} = \bar{f}(s) = \int_0^{\infty} e^{-st}f(t)dt, \quad \mathrm{Re}\, s > 0, \tag{2.2}$$

The inverse Laplace transform is

$$L^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds, \quad c > 0. \tag{2.3}$$

The Laplace transform is highly efficient for solving some class of ordinary and partial differential equations. A variety of extended Laplace-type transforms are introduced for example in [36, 37, 38, 39, 40].

2.3 Pathway-type transform

The pathway-type transform ($P_\varsigma-$transform) is given in [41, 42] in the form

$$P_\varsigma[f(w),\varphi] = F(\varphi) = \int_0^{\infty} [1 + (\varsigma - 1)\varphi]^{\frac{-w}{\varsigma-1}} f(w)dw \quad \varsigma > 1, \tag{2.4}$$

with

$$\lim_{\varsigma \to 1^+} [1 + (\varsigma - 1)\varphi]^{\frac{-w}{\varsigma-1}} = e^{-\phi w}, \tag{2.5}$$

and

$$\lim_{\varsigma \to 1^+} P_\varsigma[f(w), \varphi] = \mathrm{L}[f(w), \varphi], \tag{2.6}$$

where $(L[.,.])$ is the Laplace transform (2.2).

Some basic results of the $P_\varsigma-$ transform are given in [41] as follows

$$P_\varsigma[1, \varphi] = \frac{\varsigma - 1}{\ln[1 + (\varsigma - 1)\varphi]}, \tag{2.7}$$

$$P_\varsigma[w^\upsilon, \varphi] = \left\{ \frac{\varsigma - 1}{\ln[1 + (\varsigma - 1)\varphi]} \right\}^{\upsilon+1} \Gamma(\upsilon + 1), \quad \upsilon \in \mathbb{C}, \tag{2.8}$$

and

$$P_\varsigma[{}_0\mathbf{D}_w^{-\alpha} f(w), \varphi] = \left[\frac{\varsigma - 1}{\ln[1 + (\varsigma - 1)\varphi]} \right]^\alpha P_\varsigma[f(w), \varphi], \quad \mathrm{Re}(\alpha) > 0, \quad \varsigma > 1, \tag{2.9}$$

where ${}_0\mathbf{D}_w^{-\alpha} f(w)$ is given in (1.6).

More information about the $P_\varsigma-$transform and its applications may be found in [41, 42].

2.4 Hankel transform

The Hankel transform (also designated as Fourier-Bessel transform) is a fundamental tool in many areas of mathematical statistics, physics, engineering, probability theory, analytic number theory, data analysis, etc (see, for instance, [41, 42, 43, 44, 45, 46]). The integral

$$\mathcal{H}_\nu\{f(t); q\} = \int_0^\infty q\, \mathcal{J}_\nu(qt)\, f(t)\, dt, \;\; q > 0,\, \mathrm{R}(\nu) > \frac{1}{2}, \tag{2.10}$$

defines the Hankel transform involving the ν^{th}-order Bessel function of first kind $\mathcal{J}_\nu(\eta)$ [45] as a kernel. Its inverse transform is

$$f(t) = \mathcal{H}_\nu^{-1}\{\tilde{f}(q)\}(t) = \int_0^\infty q\mathcal{J}_\nu(tq)\tilde{f}(q)dq, \;\; t > 0. \tag{2.11}$$

2.5 Mellin transform

The Mellin integral transform is similar with the Laplace transform and Fourier transform and is widely applied in computer science and number theory due to its invariant properties [35, 36]. The Mellin transform of a suitable integrable function $f(t)$ is defined by

$$\tilde{f}(p) = \mathcal{M}\Big\{ f(t) \Big\}(p) = \int_0^\infty t^{p-1}\, f(t)\, dt, \qquad p \in \mathbb{C}, \tag{2.12}$$

provided that the improper integral in (2.12) exists. The inverse Mellin transform is

$$f(t) = \mathcal{M}^{-1}\Big\{ \tilde{f}(p) \Big\}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-p}\, \tilde{f}(p)\, dp, \tag{2.13}$$

where $c \in \backslash\{p\}$ is a constant.

Further, there are two convolution Type theorems for the Mellin transform. If $M\{f(t)\} = \tilde{f}(p)$ and $M\{g(t)\} = \tilde{g}(p)$, then

$$\begin{aligned} M[f(t) * g(t)] &= M\left[\int_0^\infty f(\xi) g\left(\frac{t}{\xi}\right) \frac{d\xi}{\xi} \right] = \tilde{f}(p)\tilde{g}(p), \\ M[f(t) \circ g(t)] &= M\left[\int_0^\infty f(t\xi) g(\xi) d\xi \right] = \tilde{f}(p)\tilde{g}(1-p). \end{aligned} \tag{2.14}$$

2.6 Beta transform

The Beta transform of $f(z)$ is defined as [48, 49]:

$$\mathbb{B}\{f(z):a,b\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz. \tag{2.15}$$

The matrix version of the Beta transform of $f(z)$ is defined as

$$\mathbf{B}\{f(z):P,Q\}=\int_0^1 z^{P-I}(1-z)^{Q-I}f(z)dz, \tag{2.16}$$

where P and Q are positive stable matrices in the complex matrix space of common order n; $\mathbb{C}^{n\times n}$.

For $f(z)=1$, we get the Beta matrix function given by Jodar and Cortés (1998) [50] as

$$\mathbb{B}\,(P,Q)=\int_0^1 t^{P-I}\ (1-t)^{Q-I}\ dt. \tag{2.17}$$

2.7 Whittaker transform

The Whittaker transform is defined in [51] as

$$\mathfrak{W}(z)=\int_0^\infty (2zt)^{\frac{-1}{4}}\,\mathbf{W}_{\lambda,\mu}(2zt)\,f(t)\,dt, \tag{2.18}$$

where$\mathbf{W}_{\lambda,\mu}(t)$ is the Whittaker function. For $\lambda=\frac{1}{4}$ and $\mu=\pm\frac{1}{4}$ the Whittaker transform goes over into the Laplace transform.

3 Fractional Kinetic Equations

The kinetic (reaction-type) equations have prime importance as mathematical tools widely used in describing several astrophysical and physical phenomena. The production and destruction of nuclei in the chemical (thermonuclear) reactions can be described by the reaction-type (kinetic) equations. Reactions characterized by a time dependent quantity $N=N(t)$ can be formally represented by the following Cauchy problem (See, Haubold and Mathai [52])

$$\frac{dN}{dt}=-\delta(N)+p(N),\ N(0)=N_0, \tag{3.1}$$

where δ and p are the destruction rate and the production rate of N, respectively, and N_0 is the initial data. Haubold and Mathai [52] studied the following special case of the Cauchy problem,

$$\frac{dN}{dt}=-\vartheta N,\ \ \vartheta\in^+,\ N(0)=N_0. \tag{3.2}$$

Equation (3.2) is known as the standard kinetic equation. An alternative form of equation (3.2) can be obtained as

$$N(t)-N_0=-\vartheta\,{}_0\mathbf{D}_t^{-1}N(t),\quad \vartheta,t\in\mathbb{R}^+, \tag{3.3}$$

where ${}_0\mathbf{D}_t^{-1}$ is the standard integral operator. Haubold and Mathai [52] have introduced a fractional generalization of the standard kinetic equation (3.3) as

$$N(t)-N_0=-\vartheta^\nu\,{}_0\mathbf{D}_t^{-\nu}N(t),\quad \vartheta,t\in\mathbb{R}^+, \tag{3.4}$$

where ${}_0\mathbf{D}_t^{-\nu}$ is defined in (1.6). The solution of the fractional kinetic equation (3.4) takes the form

$$N(t)=N_0\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\nu n+1)}(\vartheta t)^{\nu n}. \tag{3.5}$$

Further extensions and generalizations of kinetic fractional equations involving many fractional operators have grown interest in applied science not only in mathematics but also in dynamical systems, physical phenomena, engineering and control systems (see, [53, 54, 55]).

4 Related Works

Several research have been conducted to study generalizations of the SFs. This section reviews some works related to the generalized SFs associated with FCOs and ITs.

4.1 Fractional operators for the Wright hypergeometric matrix function

The generalized (Wright) hypergeometric function were first studied by Virchenko et al.(2001)[56] as follows

$$ {}_2\mathbf{R}_1^{(\tau)}(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}, \quad \tau>0, |z|<1. \tag{4.1}$$

The matrix version of (4.1) was introduced by Bakhet et al. (2019) [57] as follows,

Definition 4.1. Let A, B and C be positive stable matrices in $\mathbb{C}^{n\times n}$ such that the matrix $C+nI$ is invertible for every integer $n\geq 0$. Then, the Wright hypergeometric matrix function (WHMF) is defined in the form:

$$\begin{aligned} {}_2\mathbf{R}_1^{(\tau)}(A,B;C;z):=&\Gamma^{-1}(B)\Gamma(C)\\ &\times\sum_{n=0}^{\infty}(A)_n\Gamma^{-1}(C+\tau nI)\Gamma(B+\tau nI)\frac{z^n}{n!}, \end{aligned}\tag{4.2}$$

where $\tau\in\mathbb{R}_+=(0,\infty)$ for all integer $n\geq 0$, and

$$(A)_n=\begin{cases} A(A+I)...(A+(n-1)I)=\Gamma^{-1}(A)\Gamma(A+nI), & n\geq 1,\\ I, & n=0,\end{cases}\tag{4.3}$$

is the Pochhammer symbol of a matrix A, where

$$\Gamma(A)=\int_0^{\infty}e^{-w}\,w^{A-I}dw,\ w^{A-I}=\exp((A-I)\ln w),\tag{4.4}$$

is the Gamma matrix function (see [57]). For $\tau=1$, (4.2) reduces to the Gauss hypergeometric matrix function which is defined by Jódar and Cortés (1998) [50].

In this section, we investigate new properties of the WHMF in (4.2). Further, we introduce the Wright type hypergeometric matrix functions by using the FCOs in (1.6) and (1.7), as follows,

$$\begin{aligned}\mathcal{R}(w;A,B;\nu;\lambda)=&\frac{w^{\nu}}{\Gamma(\nu+1)}\,{}_2\mathbf{F}_1(A,B;(\nu+1)I;\lambda w)\\ &\Big(w,\nu,\lambda\in\mathbb{C}\text{ and }A,B\in\mathbb{C}^{n\times n}\Big),\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}\mathcal{R}(w;A,B;-\mu;\lambda)=&\frac{w^{-\mu}}{\Gamma(1-\mu)}\,{}_2\mathbf{F}_1(A,B;(1-\mu)I;\lambda w)\\ &\Big(w,\mu,\lambda\in\mathbb{C}\text{ and }A,B\in\mathbb{C}^{n\times n}\Big).\end{aligned}\tag{4.6}$$

• **Main Results**

Theorem 4.2. *[58]. Let A,B and C be positive stable matrices in $\mathbb{C}^{n\times n}$. Then for $\tau>0,\ m>0$, the following integrals hold true:*

$$\begin{aligned}&\int_0^\infty \exp\Big(\frac{-u^m}{z^m}\Big)\,u^{C-(\tau+1)I}\Bigg[\sum_{n=0}^{\infty}(A)_n\,\Gamma(B+\tau nI)\Gamma(C)\\&\times\Gamma^{-1}(B)\Gamma^{-1}(C+\tau nI)\Gamma^{-1}\big(\frac{C-(\tau+n)I}{m}\big)u^n\Bigg]\,du\\&=\frac{z^{C-\tau I}}{m}\,{}_2\mathbf{R}_1^{(\tau)}(A,B;C;z),\ \ |z|<1.\end{aligned}\tag{4.7}$$

$$\begin{aligned}\int_0^1 z^m\,{}_2\mathbf{R}_1^{(\tau)}(A,B;mI;z^\tau)dz=&\frac{{}_2\mathbf{R}_1^{(\tau)}(A,B;(m+1)I;1)}{m}\\&-\frac{{}_2\mathbf{R}_1^{(\tau)}(A,B;(m+2)I;1)}{m(m+1)},\quad |z^\tau|<1.\end{aligned}\tag{4.8}$$

Theorem 4.3. *[58]. Let A and B be positive stable matrices in $\mathbb{C}^{n\times n}$ such that $B+nI$ is invertible for all integer $n\geq 0$ and $|z|<1$, then*

$$\begin{aligned}&(s+1)\,{}_2\mathbf{R}_1^{(\tau)}(A,B;(s+1)I;z)-\,{}_2\mathbf{R}_1^{(\tau)}(A,B;(s+2)I;z)\\&=\Bigg\{\frac{\tau^2}{(s+2)}\}\,z^2\,\frac{d^2}{dz^2}\Big({}_2\mathbf{R}_1^{(\tau)}(A,B;(s+3)I;z)\Big)+z\frac{\tau}{(s+2)}\{\tau+2(s+1)\Bigg\}\\&\times\frac{d}{dz}\Big({}_2\mathbf{R}_1^{(\tau)}(A,B;(s+3)I;z)\Big)+s\,{}_2\mathbf{R}_1^{(\tau)}(A,B;(s+3)I;z),\quad \mathrm{Re}(s)>0.\end{aligned}\tag{4.9}$$

Theorem 4.4. *[58]. Let A and B be positive stable matrices in $\mathbb{C}^{n\times n}$ with $B+nI$ is invertible for all integer $n\geq 0$ and $\lambda,\mu\in\mathbb{C}$ such that $|\lambda w|<1,\ Re(\mu)<1$. Then*

$$\mathbf{I}^\gamma\mathcal{R}(w;A,B;\nu;\lambda)=\mathcal{R}(w;A,B;\nu+\gamma;\lambda),\tag{4.10}$$

$$\mathbf{D}^\gamma\mathcal{R}(w;A,B;\nu;\lambda)=\mathcal{R}(w;A,B;\nu-\gamma;\lambda),\tag{4.11}$$

$$\mathbf{I}^\gamma\mathcal{R}(w;A,B;-\mu;\lambda)=\mathcal{R}(w;A,B;\gamma-\mu;\lambda),\tag{4.12}$$

and

$$\mathbf{D}^\gamma\mathcal{R}_w(A,B;\nu;\lambda)=\mathcal{R}(w;A,B;-(\gamma+\mu);\lambda).\tag{4.13}$$

Theorem 4.5. *[58]. Let C be a positive stable matrix in $\mathbb{C}^{n\times n}$, The Laplace transform of $\mathcal{R}(w;-nI,C+(n-1)I;\nu;\lambda)$ and $\mathcal{R}(w;-nI,C+(n-1)I;-\mu;\lambda);\ n\in\mathbb{N}$ are given as*

$$\mathcal{L}\Big\{\mathcal{R}(w;-nI,C+(n-1)I;\nu;\lambda)\Big\}=\frac{1}{s^{\nu+1}}\,\mathcal{Y}_n(C;\lambda,-s),\tag{4.14}$$

and

$$\mathcal{L}\Big\{\mathcal{R}(w;-nI,C+(n-1)I;-\mu;\lambda)\Big\}=\frac{1}{s^{\mu-1}}\,\mathcal{Y}_n(C;\lambda,-s),\tag{4.15}$$

where $\mathcal{Y}_n(C;\lambda,-s)$ is the generalized Bessel matrix polynomial [58].

4.2 Results on the generalized hypergeometric matrix functions

The generalized Gauss and Confluent hypergeometric matrix functions are presented by Abdall et al. (2021) [60] as follows.

Definition 4.6. Assume that A, B, A^*, B^*, C^* and $C^* - B^*$ are positive stable matrices in $\mathbb{C}^{n\times n}$, such that $B^*C^* = C^*B^*$ and p be a number with $\mathrm{Re}(p) > 0$. Then, the generalized Gauss hypergeometric matrix function (GGHMF) is defined by

$$\begin{aligned} F^{(A,B)}(A^*, B^*; C^*; z; p) = \sum_{n=0}^{\infty} (A^*)_n \, \mathcal{B}^{(A,B)}(B^* + nI, C^* - B^*; p) \\ \times \mathbb{B}^{-1}(B^*, C^* - B^*) \frac{z^n}{n!}, \end{aligned} \tag{4.16}$$

and the generalized confluent hypergeometric matrix function (GCHMF) in the form

$$\begin{aligned} {}_1F_1^{(A,B)}(B^*; C^*; z; p) = \sum_{n=0}^{\infty} \mathcal{B}^{(A,B)}(B^* + nI, C^* - B^*; p) \\ \times \mathbb{B}^{-1}(B^*; C^* - B^*) \frac{z^n}{n!}. \end{aligned} \tag{4.17}$$

Founded on the generalized Beta matrix function

$$\mathcal{B}^{(A,B)}(B^*, C^*; p) = \int_0^1 t^{B^*-I} \, (1-t)^{C^*-I} \; {}_1F_1\left(A; B; \frac{-p}{t(1-t)}\right) dt. \tag{4.18}$$

and the Beta matrix function $\mathbb{B}\,(A, B)$ is given in (2.17).

Next, we give some new results for the GGHMF and GCHMF by the following theorems.

Theorem 4.7. *[60]. For the GGHMF $F^{(A,B)}(A^*, B^*; C^*; z; p)$, the following integral form holds true.*

$$\begin{aligned} &F^{(A,B)}(A^*, B^*; C^*; z; p) \\ &= \int_0^1 (1-tz)^{-A^*} t^{B^*-I} (1-t)^{C^*-B^*-I} \\ &\times \mathbb{B}^{-1}(B^*, C^* - B^*) {}_1F_1\left(A; B; \frac{-p}{t(1-t)}\right) dt, \end{aligned} \tag{4.19}$$

where $|\arg(1-z)| < \pi$.

Theorem 4.8. *[60]. For the GGHMF with $|\arg\,(1-z)| < \pi$, then the following transformation formula holds true*

$$F^{(A,B)}(A^*, B^*; C^*; z; p) = (1-z)^{-A^*} F^{(A,B)}\left(A^*, C^* - B^*; C^*; \frac{z}{z-1}; p\right).$$

Theorem 4.9. *[60]. The GGHMF $F^{(A,B)}(A^*, B^*; C^*; z; p)$ verifies the recurrence relation*

$$\begin{aligned} &\mathbb{B}(B^* + 3I; C^* - B^* + 3I)\, p \frac{d^2}{dp^2} \left[F^{(A,B)}(A^*, B^* + 3I; C^* + 6I; z; p)\right] \\ &-\mathbb{B}(B^* + 2I; C^* - B^* + 2I) B \frac{d}{dp} \left[F^{(A,B)}(A^*, B^* + 2I; C^* + 4I; z; p)\right] \\ &-\mathbb{B}(B^* + I; C^* - B^* + I)\, p \frac{d}{dp} \left[F^{(A,B)}(A^*, B^* + I; C^* + 2I; z; p)\right] \\ &+AF^{(A,B)}(A^*, B^*; C^*; z; p) = \mathbf{0}. \end{aligned} \tag{4.20}$$

Theorem 4.10. *[60]. For the GCHMF ${}_1F_1^{(A,B)}(B^*;C^*;z;p)$, the following recurrence relation holds true*

$$\mathbb{B}(B^*+3I;C^*-B^*+3I)\,p\frac{d^2}{dp^2}\left[{}_1F_1^{(A,B)}(B^*+3I;C^*+6I;z;p)\right]$$
$$-\mathbb{B}(B^*+2I;C^*-B^*+2I)B\,\frac{d}{dp}\left[{}_1F_1^{(A,B)}(B^*+2I;C^*+4I;z;p)\right]$$
$$-\mathbb{B}(B^*+I;C^*-B^*+I)\,p\frac{d}{dp}\left[{}_1F_1^{(A,B)}(B^*+I;C^*+2I;z;p)\right]$$
$$+A\,{}_1F_1^{(A,B)}(B^*;C^*;z;p)=\mathbf{0}.$$

Theorem 4.11. *[60]. The following Beta matrix transform formula holds true:*

$$\begin{aligned}&\mathbf{B}\left\{F^{(A,B)}(P+Q,B^*;C^*;yz;p):P,Q\right\}\\&=\mathbb{B}(P,Q)F^{(A,B)}(P,B^*;C^*;w;p),\end{aligned}\tag{4.21}$$

where $A,B,B^*,C^*,P,Q,P+Q$ are positive stable matrices and commuting in $\mathbb{C}^{n\times n}$ with $(\mathrm{Re}(p)\geq 0,\ |w|<1)$.

Theorem 4.12. *[60]. If $\mathrm{Re}(s)>0,\mathrm{Re}(p)\geq 0,\ M\in\mathbb{C}^{n\times n}$ and $|\ \frac{x}{s}\ |<1$, then the Laplace transform given in (2.2), holds true:*

$$\begin{aligned}&\mathcal{L}\left\{z^{M-I}F^{(A,B)}(A^*,B^*,C^*,xz;p)\right\}\\&=s^{-M}\,\Gamma(M)\,{}_1F^{(A,B)}(A^*,M,B^*;C^*;\frac{x}{s};p).\end{aligned}\tag{4.22}$$

Theorem 4.13. *[60]. If $\rho,\delta\in\mathbb{C}$, $\mathrm{Re}(p)\geq 0$ and $|\ \frac{w}{\delta}\ |<1$, then the following Whittaker transform formula (2.18) holds true,*

$$\begin{aligned}&\int_0^\infty t^{\rho-1}e^{\frac{-\delta t}{2}}W_{\lambda,\mu}(\delta t)\ F^{(A,B)}(A^*,B^*;C^*,wt;p)dt\\&=\delta^{-\rho}\frac{\Gamma(\frac{1}{2}+\mu+\rho)\Gamma(\frac{1}{2}-\mu+\rho)}{\Gamma(1-\lambda+\rho)}\times\\&\ {}_2F_1^{(A,B)}(A^*,(\frac{1}{2}+\mu+\rho)I,(\frac{1}{2}-\mu+\rho)I,B^*;C^*,(1-\lambda+\rho)I;\frac{w}{\delta};p).\end{aligned}\tag{4.23}$$

Remark 4.14. The above results improve and generalize some already known results presented by Çekim (2013) [61], Agarwal (2014) [49] and Abdalla and Bakhet (2018) [62].

4.3 Certain fractional formulas of the extended k-hypergeometric functions

In 2007, Diaz and Pariguan [63] introduced the k-hypergeometric functions as

Definition 4.15. Let $k\in\mathbb{R}^+$ and $\alpha_1,\alpha_2,y\in\mathbb{C}$ and $\alpha_3\in\mathbb{C}\setminus\mathbb{Z}_0^-$, then k-hypergeometric series is defined in the form

$${}_2H_1^k\left[\begin{array}{c}(\alpha_1;k),(\alpha_2;k)\\(\alpha_3;k)\end{array};y\right]=\sum_{n=0}^\infty\frac{(\alpha_1)_{n,k}\ (\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}.\frac{y^n}{n!},\quad |y|<\frac{1}{k},\tag{4.24}$$

where $(\cdot)_{n,k}$ is the k-Pochhammer symbol given by

$$(y)_{n,k}=\begin{cases}y(y+k)...(y+(n-1)k), & n\in\mathbb{N},\ y\in\mathbb{C}\\[2ex] 1, & n=0,k\in\mathbb{R}^+,y\in\mathbb{C}\setminus\{0\}.\end{cases}$$

Recently, numerous works have been conducted for studying the k-hypergeometric functions, for example, Asad et al. (2021)[64], Abdalla and Hidan (2021)[65], Jianrong et al. (2022)[66], and Fuli et al.(2022)[67].

Particulary, Abdalla and Hidan [65] and Hidan et al. [68] introduced and studied several properties of the following (p, k)-analogues of Gauss hypergeometric function:

$$ {}_2\mathbf{H}_1^{(p,k)}\left[\begin{array}{c}(\alpha_1;k),(\alpha_2;k)\\(\alpha_3;k)\end{array};w\right]=\sum_{j=0}^{\infty}\frac{(\alpha_1)_{j,k}\,(\alpha_2)_{j,k}}{(\alpha_3)_{j,k}}.\frac{w^j}{(pj)!}, \tag{4.25} $$

which is an entire function for $p > 1$, where $k \in \mathbb{R}^+$ and $\alpha_1, \alpha_2, w \in \mathbb{C}$ and $\alpha_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

The purpose of the next section is to continue the investigation of a new formulae like integral transforms and fractional calculus operators on the (p, k)-analogues of Gauss hypergeometric function (4.25).

- **Main Results**

Theorem 4.16. *[56]. The Laplace transform for the ${}_2\boldsymbol{H}_1^{(p,k)}$ given in (4.25) has the following form:*

$$ \begin{aligned} &\mathcal{L}\left\{\xi^{\frac{\delta}{k}-1}\,{}_2\boldsymbol{H}_1^{(p,k)}\left[\begin{array}{c}(\alpha_1,k)(\alpha_2,k)\\(\alpha_3,k)\end{array};u\xi\right]\right\}\\ &=\frac{k\Gamma^k(\delta)}{(ks)^{\frac{\delta}{k}}}\,{}_3\boldsymbol{H}_1^{(p,k)}\left[\begin{array}{c}(\alpha_1,k)(\alpha_2,k)(\delta,k)\\(\alpha_3,k)\end{array};\frac{u}{ks}\right], \end{aligned} \tag{4.26} $$

$\Big(\alpha_1, \alpha_2, u, \xi \in \mathbb{C}, \alpha_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-, \text{Re}(\alpha_1) > 0, \text{Re}(\alpha_2) > 0, \text{Re}(s) > 0,\ |\frac{u}{ks}| < 1,\ k \in \mathbb{R}^+ \textit{and}\, p \in \mathbb{N}\Big).$

Theorem 4.17. *[68]. For $\alpha_1, \alpha_2, w \in \mathbb{C}, \alpha_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ Re(\alpha_1) > 0, Re(\alpha_2) > 0, k \in \mathbb{R}^+,\ p \in \mathbb{N}$ and $0 < \beta \leq 1$, the following fractional Fourier transform (FFT) hold true:*

$$ \begin{aligned} &\mathfrak{F}_\beta\left\{{}_2\boldsymbol{H}_1^{(p,k)}\left[\begin{array}{c}(\alpha_1,k)(\alpha_2,k)\\(\alpha_3,k)\end{array};w\right]\right)\\ &=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}(\omega)^{-(\frac{n+1}{\beta})}(-1)^n(i)^{-(n+1)}\\ &\times\frac{1}{p^{p+1}(n-\frac{1}{p})(n-\frac{2}{p})..(n-\frac{p+1}{p})}. \end{aligned} \tag{4.27} $$

Theorem 4.18. *[68]. For $\alpha_1, \alpha_2, \nu, u \in \mathbb{C}, \alpha_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $\text{Re}(\alpha_1) > 0, \text{Re}(\alpha_2) > 0$, $k \in \mathbb{R}^+$, $p \in \mathbb{N}$ and $0 < \text{Re}(\nu) \leq 1$, we have*

$$ \begin{aligned} &\boldsymbol{D}_k^{\nu}\left\{u^{\frac{\delta}{k}}\,{}_2\boldsymbol{H}_1^{(p,k)}\left[\begin{array}{c}(\alpha_1,k)(\alpha_2,k)\\ \\(\alpha_3,k)\end{array};u\right]\right\}\\ &=\frac{\lambda\Gamma^k(\lambda)}{k\Gamma^k(1-\nu+\delta)}u^{\frac{1-\nu+\delta}{k}-1}\,{}_3\boldsymbol{H}_2^{(p,k)}\left[\begin{array}{c}(\alpha_1,k)(\alpha_2,k)(\delta+k,k)\\ \\(\alpha_3,k)(1-\nu+\delta,k)\end{array};u\right] \end{aligned} \tag{4.28} $$

Theorem 4.19. *[68]. Assume that $\alpha, \beta, \gamma, \delta, \eta, \vartheta, \alpha_1, \alpha_2 \in \mathbb{C}$, $\alpha_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $x > 0$, $k \in \mathbb{R}^+$ and $p \in \mathbb{N}$ such that $\text{Re}\left(\frac{\vartheta}{k}\right) > \max\left\{0, \text{Re}(\beta-\delta), \text{Re}(\alpha+\beta+\gamma-\eta)\right\}$, then we have*

$$\left(\mathbf{I}_{0,x}^{\alpha,\beta,\gamma,\delta,\eta}\, w^{\frac{\vartheta}{k}-1}\; {}_2\mathbf{H}_1^{(p,k)}\left[\begin{array}{c}(\alpha_1;k),(\alpha_2;k)\\(\alpha_3;k)\end{array};w\right]\right)(x)$$

$$=k^{\eta}\, x^{-\alpha-\beta+\eta+\frac{\vartheta}{k}-1}\,\frac{\Gamma^k(\vartheta)\Gamma^k(\vartheta-k\beta+k\delta)\Gamma^k(\vartheta-k\alpha-k\beta-k\gamma+k\eta)}{\Gamma^k(\vartheta+k\delta)\Gamma^k(\vartheta-k\alpha-k\beta+k\eta)\Gamma^k(\vartheta-k\beta-k\gamma+k\eta)}$$

$$\times\, {}_5\mathbf{H}_4^{(p,k)}\left[\begin{array}{c}(\alpha_1;k)\ (\alpha_2;k)\quad(\vartheta;k)\quad(\vartheta-k\beta+k\delta;k)\quad(\vartheta-k\alpha-k\beta-k\gamma+k\eta;k)\\ \\ (\alpha_3;k)\ (\vartheta+k\delta;k)\ (\vartheta-k\alpha-k\beta+k\eta;k)\quad(\vartheta-k\beta-k\gamma+k\eta;k)\end{array};x\right].$$

Remark 4.20. By invoking the Definitions (1.5), (1.6), (1.7), (1.8), (1.2), and (1.10), we find various special cases from Theorem 4.19.

4.4 Extended Euler's Beta-Logarithmic function

In this section, we introduce a generalization of the Euler's Beta function (EBF), which we call the extended Euler's Beta-Logarithmic function (EEBLF). Also, we discuss various properties and establish numerical comparisons between this generalization and the previous studies using MATLAB (R2018a). Furthermore, we present a new version of the Beta distribution and acquire some of its characteristics as an application in statistics.

The extended Euler's Beta-Logarithmic function (EEBLF) is defined in [69] as

$$\begin{gathered}\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]=\int_0^1 \alpha^{1-w}\,\beta^{w}\,w^{u-1}(1-w)^{v-1}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw,\\ \Big(\alpha,\beta\in\mathbb{R}^+ \text{ with } \alpha\neq\beta, \mathrm{Re}(u)>0, \mathrm{Re}(v)>0, \text{ and } \mathrm{Re}(\ell)>0\Big),\end{gathered}\tag{4.29}$$

Remark 4.21. We see certain particular cases of the $\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]$ as follows:

(i) If $\alpha=\beta=1$, then Eq (4.29) reduces to the extended Beta function (EBF) defined by Choudhary et al. (1997) [70] in the form

$$\begin{gathered}\mathrm{EB}\,(u,v;\ell)=\int_0^1 w^{u-1}(1-w)^{v-1}\exp\big(\frac{-\ell}{w(1-w)}\big)dw,\\ \Big(\mathrm{Re}(u)>0,\ \mathrm{Re}(v)>0, \text{ and } \mathrm{Re}(\ell)>0\Big).\end{gathered}\tag{4.30}$$

(ii) When $\ell=0$ in (4.29), we obtain the Beta-Logarithmic function (BLF) given by Raïssouli and Chergui (2022) [71] as

$$\begin{gathered}\mathrm{BL}_{mean}\,(\theta,\phi;\delta_1,\delta_2)=\int_0^1 \theta^{1-u}\,\phi^{u}\,u^{\delta_1-1}(1-u)^{\delta_2-1}du,\\ \Big(\mathrm{Re}(\delta_1)>0,\ \mathrm{Re}(\delta_2)>0,\ \theta,\phi\in\mathbb{R}^+ \text{ such that } \theta\neq\phi\Big).\end{gathered}\tag{4.31}$$

(iii) If $u=v=1$, then Eq (4.29) reduces to a new extension of L_{mean}, which is called the extended logarithmic mean (EL_{mean}) as

$$\begin{gathered}\mathrm{EL}_{mean}\Big[\alpha,\beta;\ell\Big]=\int_0^1 \alpha^{1-w}\,\beta^{w}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw,\\ \Big(\alpha,\beta\in\mathbb{R}^+ \text{ with } \alpha\neq\beta, \text{ and } \mathrm{Re}(\ell)>0\Big).\end{gathered}\tag{4.32}$$

(vi) If we choose $\ell = 0$ in (4.32), then we get the L_{mean} defined by

$$\mathrm{L}_{mean}\left(\theta, \phi\right) = \int_0^1 \theta^{1-x}\, \phi^x dx = \begin{cases} \frac{\theta-\phi}{\ln(\theta)-\ln(\phi)}, & \theta \neq \phi, \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{4.33}$$

(v) Taking $\alpha = \beta = 1$ and $\ell = 0$ in (4.29), we recover the classical Beta function defined in (1.4).

Some properties of the EEBLF

Theorem 4.22. *[69]. The* $\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right]$ *satisfies the following functional relation:*

$$\mathrm{EBL}\left[\alpha, \beta; u+1, v; \ell\right] + \mathrm{EBL}\left[\alpha, \beta; u, v+1; \ell\right] = \mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right].$$

Theorem 4.23. *[69]. The following inequality holds for the* $\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right]$*:*

$$\min(\alpha, \beta) \ \leq \mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right] \leq \max(\alpha, \beta),$$

$$\Big(\alpha, \beta \in \mathbb{R}^+ \textit{ with } \alpha \neq \beta, \mathrm{Re}(u) > 0, \mathrm{Re}(v) > 0, \textit{ and } \mathrm{Re}(\ell) > 0\Big).$$

Theorem 4.24. *[69]. For* $\alpha, \beta \in \mathbb{R}^+$ *such that* $\alpha \neq \beta,\ \mathrm{Re}(u) > 0,\ \mathrm{Re}(v) > 0,$ *and* $\mathrm{Re}(\ell) > 0,$ *the* $\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right]$ *satisfies the following integral representations:*

(I)

$$\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right] = 2\alpha \int_0^{\frac{\pi}{2}} \left(\frac{\beta}{\alpha}\right)^{\cos^2(\varphi)} \cos^{2u-1}(\varphi) \sin^{2v-1}(\varphi) \times \exp\left(-\ell \sec^2(\varphi) \csc^2(\varphi)\right) d\varphi,$$

(II)

$$\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right] = e^{-2\ell} \int_0^{\infty} (\alpha)^{\frac{\tau}{\tau+1}}\ (\beta)^{\frac{1}{\tau+1}} \times \frac{\tau^{v-1}}{(1+\tau)^{u+v}} \exp\left(-\ell(\tau + \tau^{-1})\right) d\tau,$$

(III)

$$\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right] = \sqrt{\alpha\beta}\, 2^{1-u-v} \int_0^{\infty} \left(\frac{\beta}{\alpha}\right)^{\frac{\tau}{2}} (1+\tau)^{u-1}(1-\tau)^{v-1} \times \exp\left(-4\ell/(1-\tau^2)\right) d\tau.$$

Theorem 4.25. *[69]. For* $\alpha,\ \beta \in \mathbb{R}^+$ *such that* $\alpha \neq \beta$, $\mathrm{Re}(u) > 0$, $\mathrm{Re}(v) > 0$*, and* $\mathrm{Re}(\ell) > 0,$ *the Mellin transform* (2.12) *of EEBLF is*

$$\mathcal{M}\left\{\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right]; \sigma\right\} = \Gamma(\sigma)\, BL_{mean}\left[\alpha, \beta; u+\sigma, v+\sigma\right].$$

Theorem 4.26. *For* $k, h \in \mathbb{N}_0$*, the following higher-order derivatives are valid for the* $\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right]$*:*

(I)

$$\frac{\partial^k}{\partial \ell^k}\left\{\mathrm{EBL}\left[\alpha, \beta; u, v; \ell\right]\right\} = (-1)^k\, \mathrm{EBL}\left[\alpha, \beta; u-k, v-k; \ell\right], \quad \mathrm{Re}(\ell) > 0,$$

(II)

$$\frac{\partial^k}{\partial\alpha^k}\left\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\right\}$$
$$=\sum_{r=0}^{\infty}\frac{1}{\alpha^k(r-k)!}(\ln(\alpha))^{r-k}\mathrm{EBL}\Big[1,\beta;u,v+r;\ell\Big],\quad \mathrm{Re}(\alpha)>0,\, r>k,$$

(III)

$$\frac{\partial^k}{\partial\beta^k}\left\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\right\}$$
$$=\sum_{r=0}^{\infty}\frac{1}{\beta^k(r-k)!}(\ln(\beta))^{r-k}\,\mathrm{EBL}\Big[\alpha,1;u+r,v;\ell\Big],\quad \mathrm{Re}(\beta)>0,\, r>k,$$

(VI)

$$\frac{\partial^{k+h}}{\partial v^h\,\partial u^k}\left\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\right\}=\int_0^1\alpha^{1-w}\,\beta^w\,w^{u-1}(1-w)^{v-1}$$
$$\times\ln^k(w)\,\ln^h(1-w)\exp\left(-\frac{\ell}{w(1-w)}\right)dw,$$
$$\left(\alpha,\,\beta\in\mathbb{R}^+\,\textit{with}\,\alpha\neq\beta,\,\mathrm{Re}(u)>0,\,\mathrm{Re}(v)>0,\,\textit{and}\,\mathrm{Re}(\ell)>0\right).$$

Numerical representations and graphs

The numerical representations of the values of the new generalizations of the logarithmic mean and Euler's Beta-Logarithmic function, besides some of its exceptional cases, are given in the form of tabulated data and graphical outcomes utilizing the MATLAB program.

Table 1. Comparison of numerical values of $\_EL_{mean}$ in (4.32) for different values for all α, β, and ℓ.

N	α	β	EL_{mean}					
			$\ell = 10^{-3}$	$\ell = 7.5 \times 10^{-4}$	$\ell = 5 \times 10^{-4}$	$\ell = 2.5 \times 10^{-4}$	$\ell = 10^{-8}$	$\ell = 0$
1	0.1	0.25	0.16121	0.16176	0.16233	0.16296	0.1637	0.1637
2	0.1	0.5625	0.2633	0.26427	0.26531	0.26642	0.26777	0.26777
3	0.1	0.875	0.35094	0.35232	0.35378	0.35537	0.3573	0.3573
4	0.1	1.1875	0.4313	0.43307	0.43495	0.437	0.43949	0.43949
5	0.1	1.5	0.50698	0.50913	0.51142	0.51392	0.51698	0.51698
6	0.825	0.25	0.47407	0.47572	0.47746	0.47935	0.48161	0.48161
7	0.825	0.5625	0.67528	0.67751	0.67985	0.68238	0.68539	0.68539
8	0.825	0.875	0.83731	0.84005	0.84294	0.84605	0.84975	0.84975
9	0.825	1.1875	0.9806	0.98383	0.98723	0.9909	0.99527	0.99527
10	0.825	1.5	1.1122	1.1159	1.1198	1.1241	1.1291	1.1291
11	1.55	0.25	0.70045	0.70307	0.70585	0.70887	0.7125	0.7125
12	1.55	0.5625	0.95925	0.96254	0.96601	0.96975	0.97423	0.97423
13	1.55	0.875	1.1629	1.1668	1.1709	1.1753	1.1805	1.1805
14	1.55	1.1875	1.3407	1.3451	1.3498	1.3548	1.3607	1.3607
15	1.55	1.5	1.5025	1.5074	1.5126	1.5182	1.5249	1.5249
16	2.275	0.25	0.9006	0.90415	0.90793	0.91203	0.91701	0.91701
17	2.275	0.5625	1.2059	1.2102	1.2147	1.2196	1.2255	1.2255
18	2.275	0.875	1.4428	1.4477	1.4529	1.4585	1.4652	1.4652
19	2.275	1.1875	1.6477	1.6532	1.659	1.6653	1.6727	1.6727
20	2.275	1.5	1.8332	1.8392	1.8456	1.8525	1.8607	1.8607
21	3	0.25	1.086	1.0905	1.0952	1.1004	1.1067	1.1067
22	3	0.5625	1.432	1.4372	1.4428	1.4488	1.4561	1.4561
23	3	0.875	1.6975	1.7035	1.7097	1.7165	1.7246	1.7246
24	3	1.1875	1.9259	1.9324	1.9394	1.9468	1.9557	1.9557
25	3	1.5	2.1316	2.1387	2.1463	2.1544	2.164	2.164

Table 2. Comparison of numerical values of $EBLF$ in (4.29) for different values for all $\alpha,\ \beta,\ u,\ v$, and ℓ.

N	$\alpha=\beta$	u	v	EBL		
				$\ell=0$	$\ell=0.40$	$\ell=0.80$
1	1	0.1	0.25	13.547	0.3861	0.058419
2	1	0.25	0.25	7.4163	0.34229	0.05213
3	1	0.4	0.25	5.8075	0.30491	0.046648
4	1	0.55	0.25	5.0329	0.27285	0.041855
5	1	0.7	0.25	4.5627	0.2452	0.037651
6	1	0.85	0.25	4.2397	0.22123	0.033953
7	1	1	0.25	4	0.20035	0.030691
8	3	0.1	0.5	33.969	0.96446	0.14647
9	3	0.25	0.5	15.732	0.84903	0.13015
10	3	0.4	0.5	11.037	0.75113	0.11597
11	3	0.55	0.5	8.8274	0.66763	0.10363
12	3	0.7	0.5	7.5174	0.59602	0.092838
13	3	0.85	0.5	6.638	0.53431	0.083383
14	3	1	0.5	6	0.48085	0.075074
15	6	0.1	0.75	62.876	1.6251	0.24658
16	6	0.25	0.75	26.657	1.421	0.21819
17	6	0.4	0.75	17.479	1.2488	0.19363
18	6	0.55	0.75	13.24	1.1028	0.17232
19	6	0.7	0.75	10.776	0.97823	0.15376
20	6	0.85	0.75	9.1543	0.87145	0.13756
21	6	1	0.75	8	0.77946	0.12337
22	9	0.1	1	90	2.0755	0.31341
23	9	0.25	1	36	1.8032	0.27622
24	9	0.4	1	22.5	1.5747	0.24415
25	9	0.55	1	16.364	1.3818	0.21642
26	9	0.7	1	12.857	1.2182	0.19236
27	9	0.85	1	10.588	1.0786	0.17142
28	9	1	1	9	0.95902	0.15315

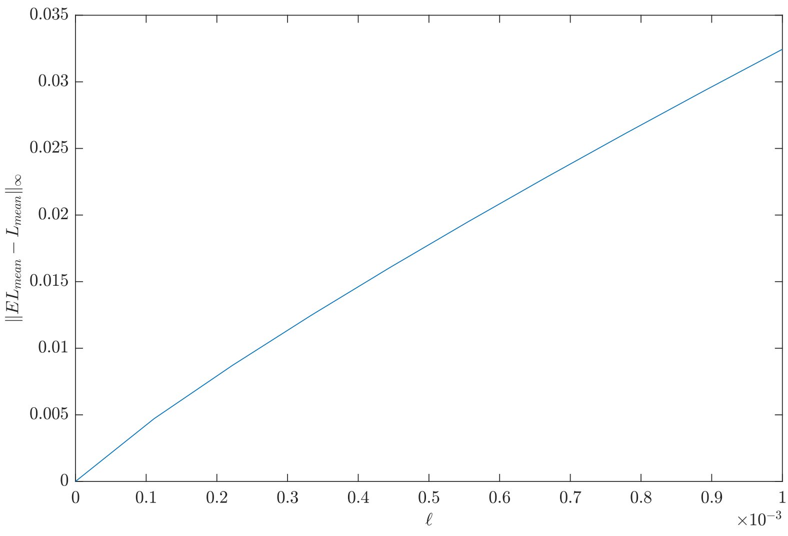


Figure 1. Graphical representation of $\|$ $\mathrm{EL}_{mean} - \mathrm{L}_{mean}$ $\|_\infty$ various values of ℓ.

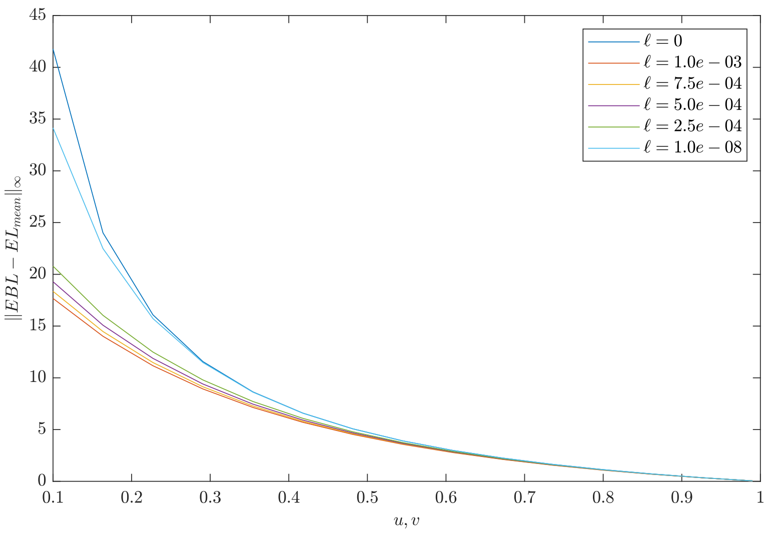


Figure 2. Plots of $\|$ $\mathrm{EBL} - \mathrm{EL}_{mean}$ $\|_\infty$ with equal values of u, v, and various values of ℓ.

Application

The extended Gauss and Kummer hypergeometric logarithmic functions of one complex variable is introduced in this section utilizing the EEBLF in (4.29), as

$$\mathbf{G}^{\ell}_{Log}\left[\begin{array}{cc}\lambda_1 & \lambda_2 \\ & \\ \lambda_3 & \end{array}\Big|\alpha_1,\alpha_2\Big|z\right] = \sum_{n=0}^{\infty} \frac{\text{EBL}\Big[\alpha_1,\alpha_2;\lambda_2+n,\lambda_3-\lambda_2;\ell\Big]}{\text{B}\Big(\lambda_2,\lambda_3-\lambda_2\Big)}(\lambda_1)_n\frac{z^n}{n!} \tag{4.34}$$

$$\Big(\ell\geq 0; |z|<1; \text{Re}(\lambda_1)>0; \text{Re}(\lambda_3)>\text{Re}(\lambda_2)>0; \alpha_1,\alpha_2\in\mathbb{R}^+ \text{ such that } \alpha_1\neq\alpha_2\Big),$$

and

$$\mathbf{K}^{\ell}_{Log}\left[\begin{array}{c}\lambda_2 \\ \\ \lambda_3\end{array}\Big|\alpha_1,\alpha_2\Big|z\right] = \sum_{n=0}^{\infty} \frac{\text{EBL}\Big[\alpha_1,\alpha_2;\lambda_2+n,\lambda_3+n;\ell\Big]}{\text{B}\Big(\lambda_2,\lambda_3-\lambda_2\Big)}\frac{z^n}{n!}, \tag{4.35}$$

$$\Big(\ell\geq 0; |z|<1; \text{Re}(\lambda_3)>\text{Re}(\lambda_2)>0; \alpha_1,\alpha_2\in\mathbb{R}^+ \text{ such that } \alpha_1\neq\alpha_2\Big),$$

respectively.

Remark 4.27. According to [70, 71] the series (4.34) is seen to converge when $|z| < 1$, provided that $\lambda_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $\ell \geq 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}^+$ such that $\alpha_1 \neq \alpha_2$.

Remark 4.28. For $\ell > 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}^+$ such that $\alpha_1 \neq \alpha_2$, the series (4.35) converges for all z, provided that $\lambda_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Additional findings on these functions are covered in [**?**].

4.5 Analytical properties of the two variables Jacobi matrix polynomials

Defez et al. (2004) [73] introduced the Jacobi matrix polynomials in the following definition.

Definition 4.29. Let E and F be positive stable matrices in $\mathbb{C}^{n\times n}$, then the Jacobi matrix polynomial (JMP) $\mathcal{P}_n^{(E,F)}(z)$ is defined by

$$\mathcal{P}_n^{(E,F)}(z) \quad = \frac{(E+I)_n}{n!}\,{}_2F_1\left[\begin{array}{c}-nI, E+F+(n+1)I \\ E+I\end{array};\frac{1-z}{2}\right]. \tag{4.36}$$

These polynomials are generalizations of several families of orthogonal matrix polynomials like the Legendre, Chebyshev and Gegenbauer (ultraspherical) matrix polynomials.

In the current section, we define and establish the 2−variable analogue of Jacobi matrix polynomials (2VA-JMP) with some properties, which have been proposed on the pattern for 2-variable Gegenbauer matrix polynomials by Kahmmash (2008) [72], 2-variable Laguerre matrix polynomials by Khan and Hassan (2010) [74], 2-variable Hermite generalized matrix polynomials by Subuhi et al. (2010)[75], and 2-variable Shivley's matrix polynomials by He et al. (2019) [76].

Definition 4.30. [77]. Let E and F be the positive stable matrices in $\mathbb{C}^{n\times n}$ such that $E + nI$ and $F + nI$ are invertible for all integers $n \geq 0$. Then $\mathcal{J}_n(E, F, z, w)$ takes the following explicit form:

$$\mathcal{J}_n(E,F,z,w) = \sum_{s=0}^{n} \frac{(E+I)_n(F+I)_n}{s!(n-s)!}[(E+I)_s]^{-1}[(F+I)_{n-s}]^{-1} \times (\frac{z-\sqrt{w}}{2})^s(\frac{z+\sqrt{w}}{2})^{n-s}, \tag{4.37}$$

where E and F satisfy the conditions $EF = FE$, and

$$Re(z) > -1 \;\forall\; z\in\sigma(E),\;\; Re(z) > -1 \;\forall\; z\in\sigma(F), \tag{4.38}$$

with $(E)_n$ denotes the matrix Pochhammer symbol.

- **Main Results**

Theorem 4.31. *[77]. Let $E, F \in \mathbb{C}^{n\times n}$ be positive stable matrices. The generating matrix function of $\mathcal{J}_n(E,F,z,w)$ is*

$$\begin{aligned}&\sum_{\nu=0}^{\infty}(E+F+I)_\nu\,[(E+I)_\nu]^{-1}\,\mathcal{J}_\nu(E,F,z,w)\,t^\nu = (1-t)^{-(I+E+F)}\\ &\times\ {}_2F_1\left[\begin{matrix}\frac{1}{2}(E+F+I),\frac{1}{2}(2I+E+F)\\ I+E\end{matrix};2t\frac{(z-\sqrt{w})}{(1-t)^2}\right],\ |t|<1,\ |\frac{(z-\sqrt{w})}{(1-t)^2}|<1.\end{aligned} \tag{4.39}$$

Theorem 4.32. *[77]. Let E and F be positive stable matrices in $\mathbb{C}^{d\times d}$ and $|\frac{t}{2}(z-\sqrt{w})|<1$ and $|\frac{t}{2}(z+\sqrt{w})|<1$, the following Bateman's generating matrix function holds true:*

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\mathcal{J}_\nu(E,F,z,w)[(E+I)_\nu]^{-1}[(F+I)_\nu]^{-1}\,t^\nu\\ &= {}_0F_1\left[\begin{matrix}-\\ E+I\end{matrix};\frac{t}{2}(z-\sqrt{w})\right]\cdot {}_0F_1\left[\begin{matrix}-\\ F+I\end{matrix};\frac{t}{2}(z+\sqrt{w})\right].\end{aligned} \tag{4.40}$$

Theorem 4.33. *[77].. Let D, E and F be commutative matrices in $\mathbb{C}^{n\times n}$ such that E and F satisfies the spectral condition (4.38) with $\left|\frac{1-\sqrt{w}t-\mathcal{R}}{2}\right|<1$ and $\left|\frac{1+\sqrt{w}t-\mathcal{R}}{2}\right|<1$, the following Brafman's generating matrix function holds true:*

$$\begin{aligned}&\sum_{\nu=0}^{\infty}(D)_\nu(E+F-D+I)_\nu\,[(E+I)_\nu]^{-1}\,[(F+I)_\nu]^{-1}\,\mathcal{J}_\nu(E,F,z,w)\,t^\nu\\ &= {}_2F_1\left[\begin{matrix}D,E+F-D+I\\ E+I\end{matrix};\frac{1-\sqrt{w}t-\mathcal{R}}{2}\right]\\ &\times\ {}_2F_1\left[\begin{matrix}D,E+F-D+I\\ F+I\end{matrix};\frac{1+\sqrt{w}t-\mathcal{R}}{2}\right],\ \mathcal{R}=(1-2zt+wt^2)^{\frac{-1}{2}}.\end{aligned}$$

Theorem 4.34. *[77]. Let E and F be matrices in $\mathbb{C}^{n\times n}$ satisfing (4.38). Then the 2VAJMP $\mathcal{J}_n(E,F,z,w)$ may be expressed as*

$$\begin{aligned}\mathcal{J}_n(E,F,z,w) &= \frac{(z-\sqrt{w})^{-E}(z+\sqrt{w})^{-F}}{2^n n!}\\ &\times\mathcal{D}^n\left[(z-\sqrt{w})^{E+nI}(z+\sqrt{w})^{F+nI}\right],\ \mathcal{D}\equiv\frac{d}{d(z\pm\sqrt{w})}\end{aligned} \tag{4.41}$$

Theorem 4.35. *[77]. The 2VAJMP $\mathcal{J}_n(E,F,z,w)$ satisfies the following differential matrix recurrence relations:*

$$\begin{aligned}&2z(E+F+nI)\,\mathbf{D}\mathcal{J}_n(E,F,z,w)\\ &+\left[z(E-F)-(E+F+2nI)\sqrt{w}\right]\mathbf{D}\mathcal{J}_{n-1}(E,F,z,w)\\ &=\sqrt{w}(E+F+nI)\left[2n\,\mathcal{J}_n(E,F,z,w)-(E-F)\,\mathcal{J}_{n-1}(E,F,z,w)\right],\end{aligned} \tag{4.42}$$

$$\begin{aligned}&2\sqrt{w}(E+F+nI)\,\mathbf{D}\mathcal{J}_n(E,F,z,w)\\ &+\left[\sqrt{w}(E-F)-(E+F+2nI)z\right]\mathbf{D}\mathcal{J}_{n-1}(E,F,z,w)\\ &=\sqrt{w}(E+F+nI)\,(E+F+2nI)\,\mathcal{J}_{n-1}(E,F,z,w),\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}&\mathbf{D}^s\,\mathcal{J}_n(E,F,z,w)\\ &=2^{-m}\,(E+F+(n+1)I)_s\ \mathcal{J}_{n-s}(E+sI,F+sI,z,w),\ 0<s\le n.\end{aligned} \tag{4.44}$$

Remark 4.36. The current results enhance and expand previous findings by Rehana et al. (2017)[78].

4.6 Hankel transforms of generalized Bessel matrix polynomials

Later on, evaluations of Hankel transform representations involving a variety of special functions and polynomials have been archived by Debnath and Bhatta (2016) [79, Chapter 7]. Also, Abdalla et al. (2021) discussed various integral transforms such as Fourier transforms [80], Laplace transforms [81], and matrix Riemann–Liouville fractional integrals [82] associated with functions involving generalized Bessel matrix polynomials.

Motivated by previous work, we give a generalization of the scalar Hankel transform (2.10) and its inverse into the matrix framework.

Definition 4.37. [83].(Matrix Hankel Transforms) Let S be a positive sable matrix in $\mathbb{C}^{n\times n}$ and let $\Phi(u)$ be a function defined for $u \geq 0$. The Hankel transform involving Bessel matrix function as kernel of $\Phi(u)$ is defined as

$$\Xi_S(v) \equiv \mathcal{H}_S\{\Phi(u); v\} \equiv \int_0^\infty \Phi(u)\, \sqrt{uv}\; J_S(uv)\, du, \tag{4.45}$$

where $v > 0$ and $J_S(uv)$ is the Bessel matrix function of the first kind defined by Jódar et al. (1994) [85].

If $\widetilde{\beta}(S) > 1/2$, Hankel's repeated integral immediately gives the inversion formula

$$\Phi(u) = \mathcal{H}_S^{-1}\{\Xi_S(v); u\} \equiv \int_0^\infty \Xi_S(v)\, \sqrt{uv}\, J_S(uv)\, dv. \tag{4.46}$$

- **Main Results**

Now, we introduce the evaluation of the Hankel matrix transforms with products of certain elementary functions and the generalized Bessel matrix polynomials $\mathcal{B}_n(u; M, N)$

$$[86].\mathcal{B}_n(u; M, N) = \sum_{m=0}^{n} \frac{(-1)^m}{m!}(-nI)_m(M + (n-1)I)_m(u\, N^{-1})^m, \tag{4.47}$$

where M and N are commuting matrices in $\mathbb{C}^{n\times n}$ such that N is an invertible matrix.

Theorem 4.38. *[86]. Let S and N be commuting matrices in $\mathbb{C}^{n\times n}$. If*

$$\Phi(u) = u^{S+(2n+\frac{1}{2})I}\, e^{-\sigma u^2}\, \mathcal{B}_n(N; (1-2n)I - S, u^2), \tag{4.48}$$

then, we have

$$\begin{aligned}\Xi_S(v) &= (2\sigma)^{-(S+(2n+1)I)}\, v^{S+(2n+\frac{1}{2})I}\, e^{\frac{-v^2}{4\sigma}} \\ &\times \mathcal{B}_n(4\sigma(I - \sigma N); (1-2n)I - S, -v^2),\end{aligned} \tag{4.49}$$

where S is a positive stable matrix in $\mathbb{C}^{n\times n}$, $\widetilde{\beta}(S + nI) > -1$, $v > 0$ and $\sigma \in \mathbb{C}$ such that $Re(\sigma) > 0$.

Theorem 4.39. *[86]. Let S, P and M be positive stable and commuting matrices in $\mathbb{C}^{n\times n}$. When*

$$\Phi(u) = u^{P+\frac{1}{2}I}\, e^{-\sigma u^2}\, \mathcal{B}_n(1; M, \sigma u^2), \tag{4.50}$$

then, we have

$$\begin{aligned}\Xi_S(v) &= (-1)^n\, \sigma^{-(\frac{1}{2}(P+S)+I)}\, (2)^{-(S+I)}\, v^{S+\frac{1}{2}I} \\ &\times \Gamma(\frac{1}{2}(P+S))\, (M + \frac{1}{2}(P+S))_n\, \Gamma^{-1}(S+I)\, [(-(\frac{1}{2}(P+S)))_n]^{-1} \\ &\times {}_2\mathbf{F}_2\left[\begin{array}{c}(1-n)I + \frac{1}{2}(P+S), nI + M + \frac{1}{2}(P+S), \\ S+I, M + \frac{1}{2}(P+S)\end{array}; -\frac{v^2}{4\sigma}\right],\end{aligned} \tag{4.51}$$

where $\widetilde{\beta}(P) > -\frac{3}{2}$, $v > 0$ and $\sigma \in \mathbb{C}$ such that $Re(\sigma) > 0$.

Theorem 4.40. *[86]. Let $\mathcal{B}_n(u; M, N)$ be given in (4.47). If*

$$\Phi(u) = \mathcal{B}_n(\sigma u^2; M, N), \tag{4.52}$$

then, we have

$$\begin{aligned}\Xi_S(v) =& 2^{\frac{1}{2}}\, v^{-1}\, \Gamma(\frac{1}{2}S + \frac{3}{4}I)\, \Gamma^{-1}(\frac{1}{2}S + \frac{1}{4}I) \\ &\times {}_4\mathbf{F}_0 \left[\begin{array}{c} -nI, M + (n-1)I, \frac{1}{2}S + \frac{3}{4}I, \frac{-1}{2}S + \frac{3}{4}I, \\ ; 4\sigma(v^2 N)^{-1} \\ - \end{array} \right],\end{aligned} \tag{4.53}$$

where $\widetilde{\beta}(S) > -1/2$ *and* $v > 0$.

Theorem 4.41. *[86]. Let* $\mathcal{B}_n(u; M, N)$ *be given in* (4.47)*. When*

$$\Phi(u) = \log u\, \mathcal{B}_n(\lambda u^2; M, N), \tag{4.54}$$

then, we have

$$\begin{aligned}\Xi_S(v) &= \frac{1}{v\sqrt{2}}\, \Gamma(\frac{1}{2}S + 3/4I)\, \Gamma^{-1}(\frac{1}{2}S + 1/4I) \\ &\times \sum_{m=0}^{n} (-nI)_m\, (M + (n-1)I)_m \\ &\times (\frac{1}{2}S + 3/4I)_m\, (\frac{-1}{2}S + 3/4I)_m\, \frac{(4\lambda(Nv^2)^{-1})^m}{m!} \\ &\times \left\{ \psi(\frac{1}{2}S + (3/4 + m)I) + \psi(\frac{1}{2}S + (1/4 - m)I) - \log(v^2/4) \right\}\end{aligned} \tag{4.55}$$

where M, N *and* S *are commuting matrices in* $\mathbb{C}^{n\times n}$ *such that* $\widetilde{\beta}(S) > -3/2$, $\psi(S)$ *is the digamma matrix function defined by*

$$\psi(S) = \Gamma^{-1}(S)\Gamma'(S),$$

where $\Gamma^{-1}(S)$ *and* $\Gamma'(S)$ *are reciprocal and derivative of the Gamma matrix function, respectively and* $v > 0,\ \lambda > 0$.

Remark 4.42. Abdalla et al. (2021)[56] work into the Hankel matrix transform revealed further features and applications such as the convolution property and solve of partial differential equations of second order.

4.7 Applications.

Nowadays, an interesting application of special functions in allied sciences is solving generalized fractional kinetic equations by using various integral transforms, including the recent papers by Agarwal et al. (2018) [80], Singh et al. (2021) [85], Abdalla and Akel (2022) [83], Hidan et al. (2022) [89], Chand et al. (2024) [88], and Alqarni (2024) [89]. Here, we will show two applications to fractional kinetic equations.

Application of Mellin integral transform in solving generalized kinetic equations involving Hadamard fractional operators

Here, we introduce the solution to generalized fractional kinetic equations involving Hadamard fractional integral and the generalized extended k-Hurwitz-Lerch ζ-matrix functions.

The generalized extended k-Hurwitz-Lerch $\zeta-$ matrix function is defined by

Definition 4.43. [82]. Let T, D, E and F be positive stable matrices in $\mathbb{C}^{m\times m}$, such that $T+\ell I$ and $F+\ell I$ are invertible for all $\ell\in\mathbb{N}_0$, $\sigma\in\mathbb{R}_0^+$, $k\in\mathbb{R}^+$, and $\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then, for $|w|<1$, the generalized extended k-Hurwitz-Lerch $\zeta-$ matrix function is defined by:

$$ {}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{array}{c} D,E \\ F \end{array};w\right]=\sum_{n=0}^{\infty}(n+\alpha)^{-T}\,(D;\sigma)_{n,k}\,(E)_{n,k}\,[(F)_{n,k}]^{-1}\,\frac{w^n}{n!}, \tag{4.56}$$

where $(D;\sigma)_{n,k}$ is the generalized k-Pochhammer matrix symbols defined as

$$(D;\sigma)_{n,k}=\begin{cases}\Gamma_k^{\sigma}(D+nI)\,\Gamma_k^{-1}(D), & (\widetilde{\mu}(D)>0,\sigma,k\in\mathbb{R}^+,n\in\mathbb{N})\\ (D)_{n,k}, & (\sigma=0,\ k\in\mathbb{R}^+,n\in\mathbb{N})\\ I, & (n=0,\ \sigma=0,\ k=1)\end{cases} \tag{4.57}$$

Lemma 4.44. *[83]. The Mellin transform of the extended k-Hurwitz-Lerch $\zeta-$ matrix function is given by*

$$\begin{aligned}&\mathcal{M}\left\{{}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{array}{c} D,E \\ F \end{array};w\right]:\sigma\to\delta\right\}\\ &=\Gamma_k(\delta)\,(D)_{\delta,k}\ {}_2\Theta_1^{T;k,\alpha;\sigma}\left[\begin{array}{c} D+\delta I,E \\ F \end{array};w\right],\end{aligned} \tag{4.58}$$

where $\Re(\delta)>0$ and $\widetilde{\mu}(D+\delta I)>0$ when $\sigma=0$ and $k=1$.

where $\Re(\varepsilon)>0$ and $\tilde{\mu}(D+\varepsilon I)>0$ when $\rho=0$ and $k=1$.

$$\left({}_HI^{\gamma}f\right)(t)=\frac{1}{\Gamma(\gamma)}\int_t^{\infty}\left(\log\frac{\tau}{t}\right)^{\gamma-1}\frac{f(\tau)}{\tau}d\tau, t>0,\ \mathrm{Re}(\gamma)>0.$$

Lemma 4.45. *[83]. If $\mathrm{Re}(\gamma)>0$, $\tau\in\mathbb{C}$, and the Mellin transform $\mathcal{M}(f)(\tau)$ exists for a function f, then the following hold true*

$$\mathcal{M}\left({}_HI_+^{\gamma}f\right)(\tau)=(-\tau)^{-\gamma}\left(\mathcal{M}f\right)(\tau),\ \ \mathrm{Re}(\tau)<0,$$

and

$$\mathcal{M}\left({}_HI_-^{\gamma}f\right)(\tau)=(\tau)^{-\gamma}\left(\mathcal{M}f\right)(\tau),\ \ \mathrm{Re}(\tau)>0.$$

where ${}_HI_+^{\gamma}f$ and ${}_HI_-^{\gamma}f$ are the Hadamard fractional integral operators of order $\gamma\in\mathbb{C}$ defined in (1.12).

- **Main Results**

Theorem 4.46. *[82]. Let T_μ, D_μ, E_μ, F_μ and C be positive stable matrices in $\mathbb{C}^{m\times m}$, such that $T_\mu+\ell I$ and $F_\mu+\ell I$ are invertible for all $\mu\in\mathbb{N}$, $\ell\in\mathbb{N}_0$, $\delta,\sigma\in\mathbb{R}_0^+$, $d,k,\xi\in\mathbb{R}^+$, and $\alpha_\mu\in\mathbb{C}\backslash\mathbb{Z}_0^-$. Then, for $\gamma\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $t\in[0,\xi]$ and ${}_2\Theta_1^{T_\mu;k_\mu,\alpha_\mu;\sigma}$ is generalized of (2.6), the generalized fractional kinetic matrix equation*

$$N(t)I-N_0t^{\delta-1}\prod_{\mu=1}^{n}{}_2\Theta_1^{T_\mu;k_\mu,\alpha_\mu;\sigma}\left[\begin{array}{c} D_\mu,E_\mu \\ F_\mu \end{array};d^{\gamma}t^{\gamma}\right]=-C^{\gamma}{}_HI_t^{\gamma}N(t) \tag{4.59}$$

is solvable. The solution to (4.59) is given by

$$\begin{aligned} N(t)I =&N_0\,\xi^{\delta-1}\log(t)\prod_{\mu=1}^{n}\sum_{s=0}^{\infty}(s+\alpha_\mu)^{-T_\mu}(D_\mu;\sigma)_{n,k_\mu}(E_\mu)_{n,k_\mu}[(F_\mu)_{n,k_\mu}]^{-1}\\ &\times\left(\frac{d^{\gamma s}\xi^{\gamma s}}{s!}\right)^{\mu}\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\left[-\left(\log t^{C}\right)^{\gamma}\right]^{r}\left[\log t^{-(\gamma\mu s+\delta-1)}\right]^{\ell}\Gamma[1-(\gamma r+\ell+2)]. \end{aligned} \tag{4.60}$$

A number of exceptional instances of this result are provided in [82].

Application of pathway-type integral transform in solving fractional $\lambda-$kinetic equations involving the generalized degenerate hypergeometric functions

In the current section, we propose a new solution to fractional $\lambda-$kinetic equations based on generalized degenerate hypergeometric functions (GDHFs), which was further supported by graphical presentations derived through pathway-type transform approach methodologies.

Recently, Yağci and Şahin [85] introduced the degenerate Pochhammer symbol using the degenerate Gamma function as follows:

$$\begin{aligned} (\varpi;\lambda)_\ell &= \frac{\Gamma_\lambda(\varpi+\ell)}{\Gamma(\varpi)}\\ &= \frac{1}{\Gamma(\varpi)}\int_0^\infty (1+\lambda w)^{-\frac{1}{\lambda}}w^{\varpi+\ell-1}\,dw, \quad \lambda>\mathrm{Re}(\varpi+\ell)>0, \end{aligned} \tag{4.61}$$

where $\lambda\in(0,1)$ and $\lim_{\lambda\to 0}(\varpi;\lambda)_\ell = (\varpi)_\ell$, is the standard Pochhammer symbol. By using (4.61), the GDHF is defined in [**?**] as

$${}_m\mathbb{DH}_n^{\lambda}(w) = {}_m\mathbb{DH}_n^{\lambda}\left[\begin{array}{cc} (\gamma_1;\lambda)\cdots\gamma_m & \\ & ;w\\ \vartheta_1\cdots\vartheta_n & \end{array}\right] = \sum_{r=0}^{\infty}\frac{(\gamma_1;\lambda)_r\cdots(\gamma_m)_r}{(\vartheta_1)_r\cdots(\vartheta_n)_r}.\frac{w^r}{r!}, \tag{4.62}$$

where $w,\ \gamma_i\in\mathbb{C}$ for $i=1,2,3,...,m$, and $\vartheta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$ for $j=1,2,3,...,n$.

- **Main Results**

Theorem 4.47. *[89]. Let $\alpha,\beta,\sigma,\wp\in\mathbb{R}^+, w\in\mathbb{C},\lambda\in(0,1),\gamma_i\in\mathbb{C}$ for $i=1,2,3,...,m$ and $\vartheta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$ for $j=1,2,3,...,n$. The solution of*

$$\mathcal{K}(w)-\mathcal{K}_0\,{}_m\mathbb{DH}_n^{\lambda}(\sigma^\beta w^\beta) = -\wp^{\alpha}\,{}_0\mathbb{D}_w^{-\alpha}\mathcal{K}(w), \tag{4.63}$$

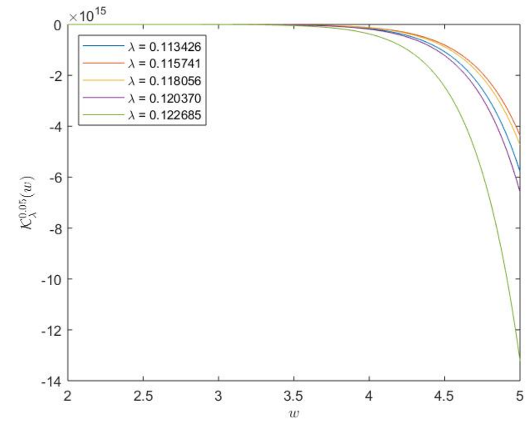
is

$$\mathcal{K}_\lambda^{\alpha}(w) = \mathcal{K}_0\sum_{r=0}^{\infty}\frac{(\gamma_1;\lambda)_r\cdots(\gamma_m)_r}{(\vartheta_1)_r\cdots(\vartheta_n)_r}\frac{\Gamma(\beta r+1)}{r!}\sigma^{\beta r}\,w^{r\beta}\,\mathrm{E}_{\alpha,\beta r+1}(-\wp^{\alpha}\,w^{\alpha}), \tag{4.64}$$

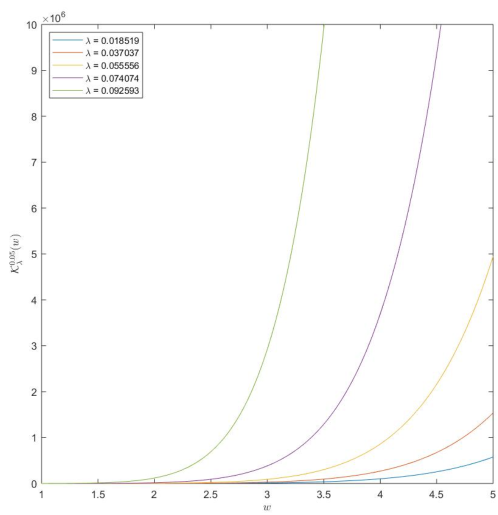
where $\mathrm{E}_{\theta,\vartheta}(\eta)$ is the generalized Mittag-Leffler function defined as

$$E_{\theta,\vartheta}(\eta) = \sum_{\imath=0}^{\infty}\frac{\eta^{\imath}}{\Gamma(\imath\theta+\vartheta)}\quad (\theta,\vartheta\in\mathbb{C},\ \mathrm{Re}(\theta)>0,\ \mathrm{Re}(\vartheta)>0). \tag{4.65}$$

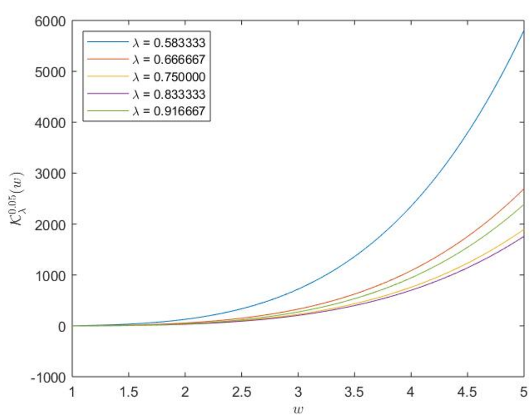
A variety of special cases of this result are given in [89].



(1A)

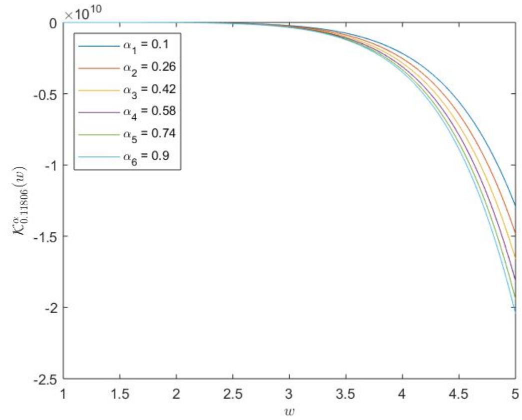


(1B)

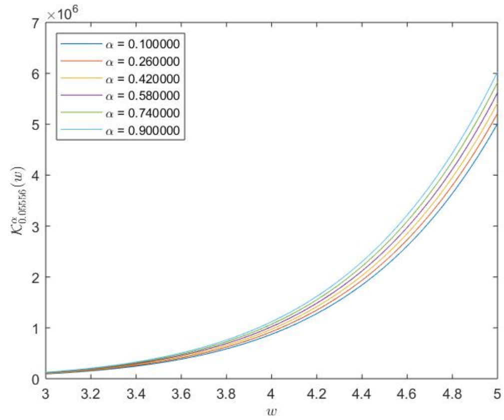


(1C)

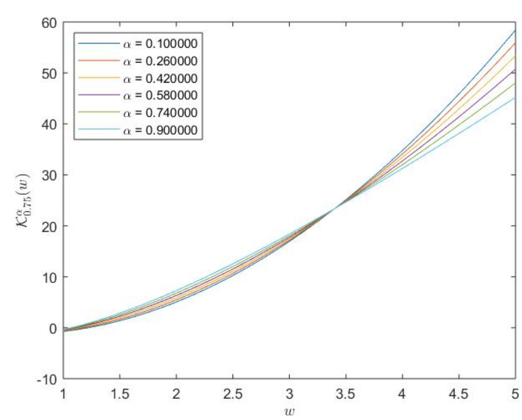
Figure 3. Solutions of (4.64) for $\mathcal{K}_{\lambda}^{\alpha}(w)$ with different values of λ in ($\mathbf{1A}$˘$\mathbf{1C}$).



(2A)



(2B)



(2C)

Figure 4. Solutions of (4.64) for $\mathcal{K}^{\alpha}_{\lambda}(w)$ with different values of α in ($\mathbf{2A}$-$\mathbf{2C}$).

5 Concluding Remarks and Future Works

The applications of special functions and fractional analysis theory are not restricted to the issues listed above. Recently, the study of the extension of several special functions and polynomials linked with fractional operators has gained importance. Thus, the researchers have derived several generalizations and developments of special functions and fractional calculus through various studies. From this perspective, in this review article, we offer a variety of results from these investigations.

As future work, we are, joining with others, planning to:

- Extend the results given in the above sections to quaternionic and Clifford analysis.
- Investigate the numerical methods for special matrix functions related to fractional calculus.
- The applications of analytical properties of generalized special functions of fractional calculus across different fields are studied.

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