Karshi Multidisciplinary International Scientific Journal

Vol. 1(1) (2024), 27–40

FRACTALS AND ITS APPLICATIONS

D.A. Juraev and N.M. Mammadzada

Communicated by Juan Bulnes

Abstract: In this article, we are going to take a detailed look at the definitions of fractals, applications of fractals, and others. Fractals are complex geometric shapes that exhibit self-similarity at different scales. They are created by repeating a simple process over and over again, resulting in complex and fascinating patterns. Fractals have found many applications in various fields due to their unique properties and visual appeal.

Keywords and phrases: Fractals, geometric fractals, algebraic fractals, stochastic fractals, initiators and generators.

MSC 2010 Classifications: Primary 35J46; Secondary 35J56.

1 Introduction

People encounter fractals everywhere. They are found in living and inanimate nature and are used in science and technology. The properties of fractals were discovered by scientists not so long ago. Fractals are a new world of beautiful shapes. By studying them, a person tries to construct the geometry of things that do not have geometry. For example, build a mathematical model of the same fern leaf. A new direction is being developed thanks to the mathematician Benoit Mandelbrot, a man who changed geometry and provoked a revolution in many areas of science, industry and art. He owns the most famous fractal - the Mandelbrot set. In it, the smallest parts are very similar to the whole, self-similarity goes infinitely deep, each pattern consists of a smaller copy of itself.

Until recently, geometric models of various natural structures were traditionally built on the basis of relatively simple geometric figures: straight lines, polygons, circles, polyhedra, spheres. However, it is obvious that this classical set, quite sufficient for describing elementary structures, becomes poorly applicable to characterize such complex objects as the outline of continental coastlines, the velocity field in a turbulent fluid flow, a lightning discharge in the air, porous materials, the shape of clouds, snowflakes, the flame of a fire, the contours of a tree, the human circulatory system, the surface of a cell membrane, etc. In the last 15-20 years, scientists are increasingly using new geometric concepts to describe these and similar formations. One of these concepts, which changed many traditional ideas about geometry, was the concept of a fractal. It was introduced by the remarkable French mathematician of Polish origin Benoit Mandelbrot in 1975. And although similar constructions in one form or another appeared in mathematics many decades ago, in physics the value of such ideas was realized only in the 70s of our century. An important role in the widespread dissemination of the ideas of fractal geometry was played by B. Mandelbrot's wonderful book "Fractal Geometry of Nature" [2]. Fractal objects, according to their initial definition, have a dimension that strictly exceeds the topological dimension of the elements from which they are built. Describing new ideas, Mandelbrot wrote:

"Why is geometry often called cold and dry? One reason is its inability to describe the shape of a cloud, mountain, tree or seashore. Clouds are not spheres, mountains are not cones, coastlines are not circles, and the crust is not smooth, and lightning does not travel in a straight line. Nature shows us not just a higher degree, but a completely different level of complexity. The number of different length scales in structures is always infinite."

The basis of the new geometry is the idea of self-similarity. It expresses the fact that the hierarchical principle of organization of fractal structures does not undergo significant changes when viewed through a microscope with different magnifications. As a result, these structures on small scales look, on average, the same as on large scales. A distinction must be made here between Euclidean geometry, which deals exclusively with smooth curves, and infinitely rugged, self-similar fractal curves. Elements of curves in Euclid are always self-similar, but in a trivial way: all curves are locally straight, and a straight line is always self-similar. A fractal curve, ideally, on any scale, even the smallest, does not reduce to a straight line and is, in the general case, geometrically irregular and chaotic. For it, in particular, there is no concept of a tangent at a point, since the functions describing these curves are, in the general case, non-differentiable.

In the past, mathematics has been concerned largely with sets and functions to which the methods of classical calculus can be applied. Sets or functions that are not sufficiently smooth or regular have tended to be ignored as 'pathological' and not worthy of study. Certainly, they were regarded as individual curiosities and only rarely were thought of as a class to which a general theory might be applicable.

The concepts of fractal and fractal geometry have become firmly established in the everyday life of mathematicians and programmers, as well as other researchers, since the mid-80s. The word fractal is derived from the Latin words: fractus - broken, shattered, fractional and the corresponding verb frangere - to break, break, that is, to create fragments of irregular shape. The birth of fractal geometry is usually associated with the publication of Mandelbrot B.'s book "Fractal Geometry of Nature" in 1977.

A fractal is a structure consisting of parts that are in some sense similar to the whole. This definition contains a significant distinguishing feature - a fractal looks the same, no matter what scale we observe it at. But, having only the appearance, the assessment of fractal properties is difficult, and in most cases impossible.

Basic properties of fractal sets:

1. the fractal has a fine structure, that is, it contains arbitrarily small scales;
2. a fractal is too irregular to be described in traditional geometric language;
3. the fractal has a form of self-similarity (approximate or statistical);
4. fractal dimension is greater than topological dimension;
5. In most cases, a fractal is defined very simply, for example, recursively.

One of the main properties that unites all fractals is the geometric repetition of itself at any scale level (self-similarity). In other words, a self-similar object is exactly or approximately the same as a part of itself, that is, the whole has the same shape as one or more parts. Self-similarity (symmetry) – invariance under parallel translations and scaling (scale changes). In the simplest case, a small part of a fractal contains information about the entire fractal. Most natural objects exhibit self-similarity, the main organizing principle of fractals. You can learn about fractals and its applications in works [1]-[8]. Using the properties of fractals, various problems were studied and solved in works [9]-[12].

Where do fractals occur in life? These phenomena, in addition to mathematicians, are observed by the natural sciences - physics and biology. The principle of fractals is used in radio engineering and to create new electronic communicators. Fractals make the operation of computer networks as stable as possible. Where do fractals occur in nature? Fractals, like patterns and shapes that repeat themselves on different scales, are found in living and inanimate nature. These are trees, rivers, mountains, plants, systems of living organisms and the structures of the Universe.

We begin by looking briefly at a number of simple examples of fractals, and note some of their features. The middle third Cantor set is one of the best known and most easily constructed fractals; nevertheless it displays many typical fractal characteristics. It is constructed from a unit interval by a sequence of deletion operations; see Figure 1. Let E_0 be the interval $[0,1]$. (Recall that $[a,b]$ denotes the set of real numbers x such that $a \leq x \leq b$). Let E_1 be the set obtained by deleting the middle third of E_0, so that E_1 consists of the two intervals $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$. Deleting the middle thirds of these intervals gives E_2; thus E_2 comprises the four intervals $\left[0,\frac{1}{9}\right]$, $\left[\frac{2}{9},\frac{1}{3}\right]$, $\left[\frac{2}{3},\frac{7}{9}\right]$, $\left[\frac{8}{9},1\right]$. We continue in this way, with E_k obtained by deleting the middle third of each interval in E_{k-1}. Thus E_k consists of 2^k intervals each of length 3^{-k}. The middle third Cantor set F consists of the numbers that are in E_k for all k; mathematically, F is the intersection $\bigcap_{k=0}^{\infty} E_k$. The Cantor set F may be thought of as the limit of the sequence of sets E_k as k tends to infinity. It is obviously impossible to draw the set F itself, with its infinitesimal detail, so 'pictures of F' tend to be pictures of one of the E_k, which are a good approximation to F when k is reasonably large; see Figure 1. At first glance it might appear that we have removed so much of the interval $[0, 1]$ during the construction of F, that nothing remains. In fact, F is an infinite (and indeed uncountable) set, which contains infinitely many numbers in every neighbourhood of each of its points.

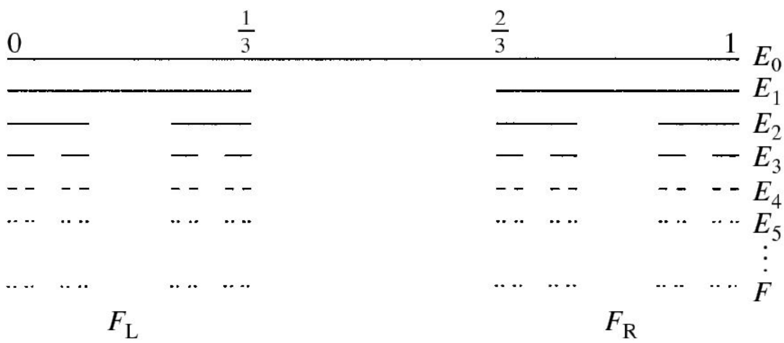


Figure 1. Construction of the middle third Cantor set F, by repeated removal of the middle third of intervals. Note that F_L and F_R, the left and right parts of F, are copies of F scaled by a factor $\frac{1}{3}$.

The middle third Cantor set F consists precisely of those numbers in $[0, 1]$ whose base-3 expansion does not contain the digit 1, i.e. all numbers $a_1 3^{-1} + a_2 3^{-2} + a_3 3^{-3} + ...$ with $a_i = 0$ or 2 for each i. To see this, note that to get E_1 from E_0 we remove those numbers with $a_1 = 1$, to get E_2 from E_1 we remove those numbers with $a_2 = 1$, and so on. We list some of the features of the middle third Cantor set F; as we shall see, similar features are found in many fractals.

2 Types of fractals

Fractals are mathematical shapes that split into smaller versions of themselves, and the parts become smaller copies of the whole. Sometimes they are created by continuously repeating a basic procedure in an endless loop or series. The property of self-similarity is central to fractals.

Fractals are usually divided into geometric, algebraic and stochastic.

Geometric — are built on the basis of the original figure, which is divided and transformed in a certain way at each iteration.

Algebraic — are built on the basis of algebraic formulas.

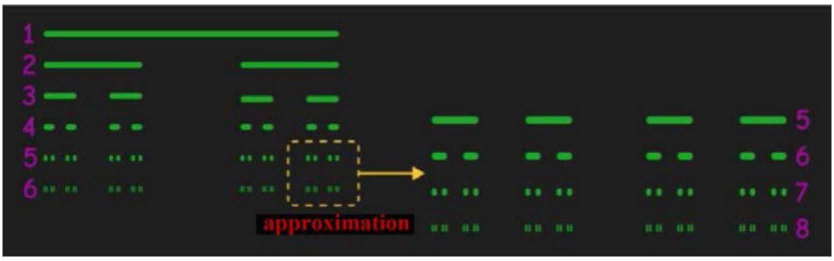
Stochastic — are formed if one or more parameters are randomly changed in the iteration system.

Next, we will examine each class in detail.

Geometric fractals. These figures are based on straight lines, squares, circles, polygons, and polyhedrons. Let's look at several examples from the simplest to the most complex.

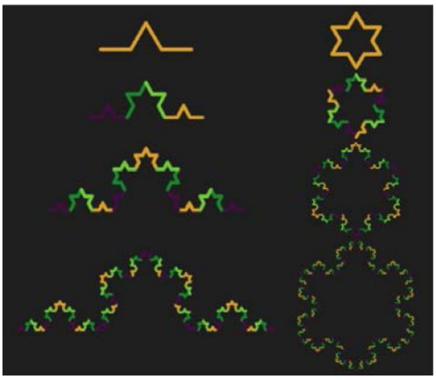
Cantor set. In 1883, Georg Cantor, a German mathematician and the author of set theory, came up with a set that repeated itself over and over again. Cantor took an arbitrary segment and divided it into two parts, then each part into two more, and so on:

Each stage of dividing the lines into two parts is called an iteration. An iteration is a repetition of the same action, or, by analogy with programming, one pass through the body of the cycle. In the first iteration we had one segment, in the second we got two, in the third - four, and so on. If we repeat this simple action an infinite number of times and zoom in on the image, we will see the same picture as at the very beginning. This is the visual embodiment of self-similarity:



Swedish mathematician Helge von Koch described the curve in 1904 using a triangle and the method of self-similarity, resulting in a fractal snowflake.

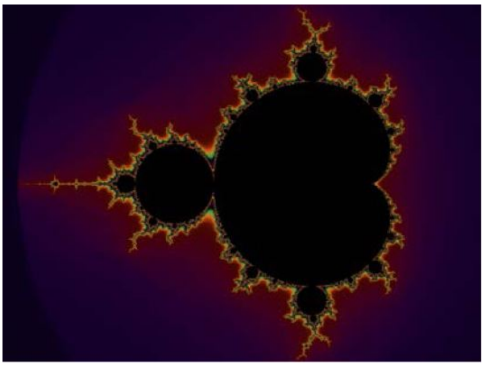
Below are four iterations of constructing such a figure. On the left are the original curves, and on the right is the snowflake obtained from these curves. It is easy to see that both the equilateral triangle and the curve itself fit perfectly into the snowflakes:



No matter what iteration we zoom in on, we can always see a familiar pattern, just like with the Cantor set. It is impossible to calculate the perimeter of such a snowflake, because it can grow further and further... This is another property of fractals - infinity.

Algebraic fractals. Algebraic fractals, unlike geometric ones, are based on a formula rather than on figures, but they are also recursively iterated. They look even more bizarre than those we have considered above.

Mandelbrot set. In 1905, the French mathematician Pierre Fatou described a set that was first modeled in the 1970s by Benoit Mandelbrot (we are already familiar with him) using a computer on the complex plane:

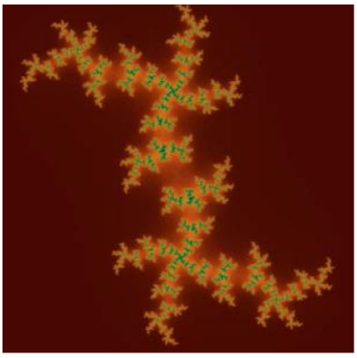


The basis of such a set is the formula:

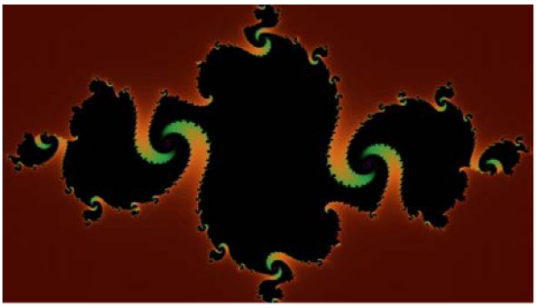
$$z_{n+1} = z_n + \mathbb{C},$$

where z and $\mathbb{C}$ are complex numbers.

Julia set. The set created by the Frenchman Gaston Julia is also based on the Fatou formula $f(z) = z^2 + \mathbb{C}$ and resembles the Mandelbrot fractal, but has some mathematical differences that affect the final result:

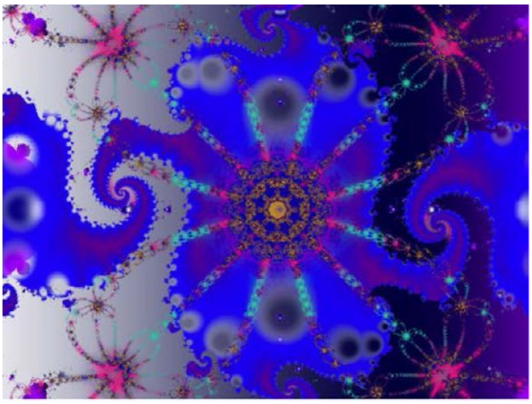


Without going into mathematical subtleties, we can say that the Mandelbrot set uses a new value of the parameter $\mathbb{C}$ at each iteration, while Julia left this value fixed at each new cycle. Therefore, for different values of $\mathbb{C}$, the Julia fractal can be visualized in different ways, for example like this:



Stochastic fractals. If in geometric and algebraic fractals the formula is constant, then in stochastic fractals it changes - and more than once. The change can occur either according to a specific law or randomly, but in both cases it leads to a fantastic visual effect!

The following image is based on several fractal formulas:

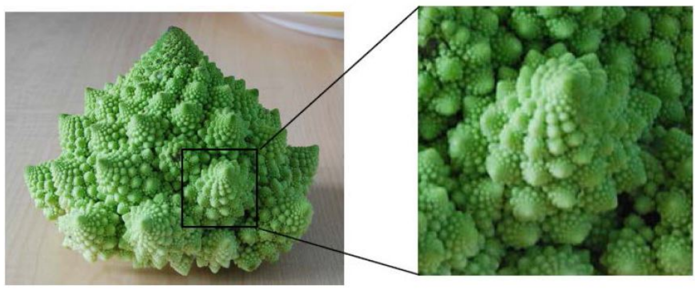


Using complex stochastic laws, scientists can reproduce the structures of living objects. By adding deviations at various iterations to fractals such as the Pythagorean tree or the Koch snowflake, we can obtain an image of tilted foliage or generate as many unique snowflakes as we like.

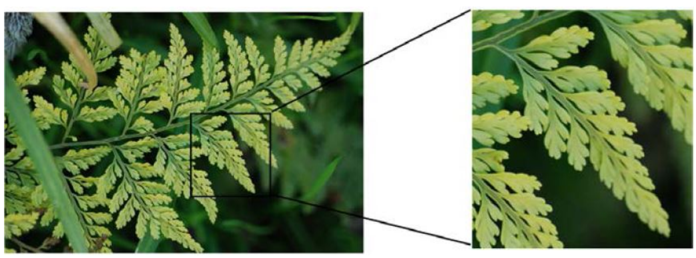
3 Examples of fractals

Fractals are mathematical sets, usually obtained through recursion, that exhibit interesting dimensional properties. For now, we can begin with the idea of self-similarity, a characteristic of most fractals. A shape is self-similar when it looks essentially the same from a distance as it does closer up.

Self-similarity can often be found in nature. In the Romanesco broccoli pictured below, if we zoom in on part of the image, the piece remaining looks similar to the whole.



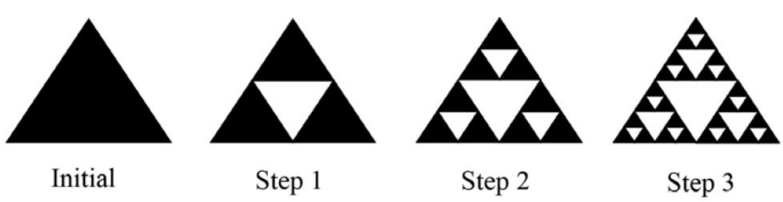
Likewise, in the fern frond below, one piece of the frond looks similar to the whole.



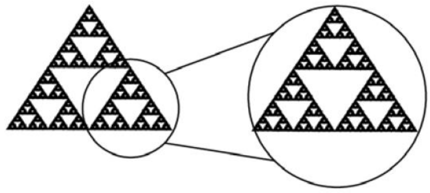
Similarly, if we zoom in on the coastline of Portugal, each zoom reveals previously hidden detail, and the coastline, while not identical to the view from further way, does exhibit similar characteristics.

This self-similar behavior can be replicated through recursion: repeating a process over and over.

Example 1. Suppose that we start with a filled-in triangle. We connect the midpoints of each side and remove the middle triangle. We then repeat this process.



If we repeat this process, the shape that emerges is called the Sierpinski gasket. Notice that it exhibits self-similarity – any piece of the gasket will look identical to the whole. In fact, we can say that the Sierpinski gasket contains three copies of itself, each half as tall and wide as the original. Of course, each of those copies also contains three copies of itself.



We can construct other fractals using a similar approach. To formalize this a bit, we're going to introduce the idea of initiators and generators.

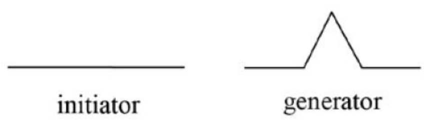
Initiators and Generators. An initiator is a starting shape A generator is an arranged collection of scaled copies of the initiator.

To generate fractals from initiators and generators, we follow a simple rule:

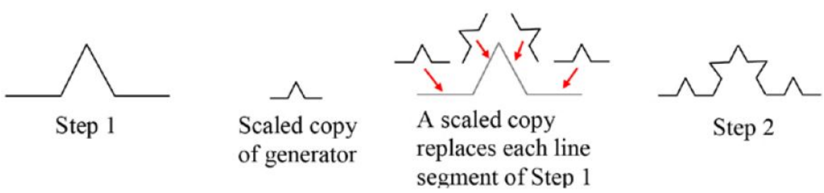
At each step, replace every copy of the initiator with a scaled copy of the generator, rotating as necessary.

This process is easiest to understand through example.

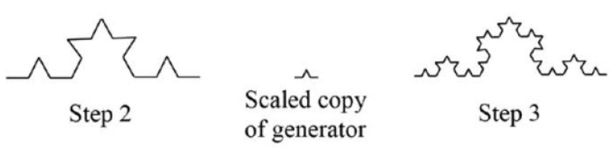
Example 2. Use the initiator and generator shown to create the iterated fractal.



This tells us to, at each step, replace each line segment with the spiked shape shown in the generator. Notice that the generator itself is made up of 4 copies of the initiator. In step 1, the single line segment in the initiator is replaced with the generator. For step 2, each of the four line segments of step 1 is replaced with a scaled copy of the generator:



This process is repeated to form Step 3. Again, each line segment is replaced with a scaled copy of the generator.



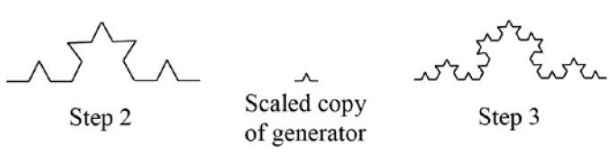
Notice that since Step 0 only had 1 line segment, Step 1 only required one copy of Step 0.

Since Step 1 had 4 line segments, Step 2 required 4 copies of the generator.

Step 2 then had 16 line segments, so Step 3 required 16 copies of the generator.

Step 4, then, would require 16*4 = 64 copies of the generator.

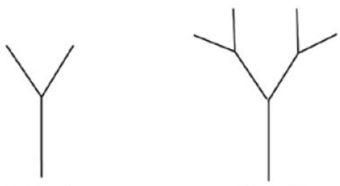
The shape resulting from iterating this process is called the Koch curve, named for Helge von Koch who first explored it in 1904.



Example 3. Use the initiator and generator below, however only iterate on the "branches." Sketch several steps of the iteration.

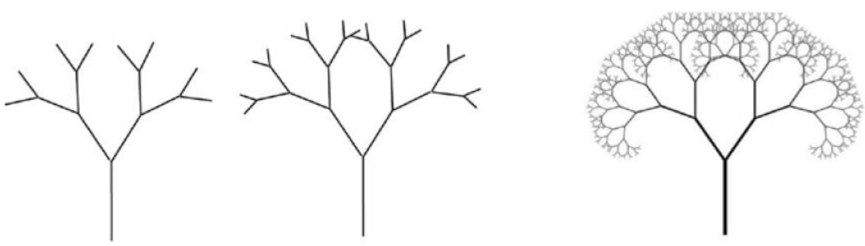
initiator generator

We begin by replacing the initiator with the generator. We then replace each "branch" of Step 1 with a scaled copy of the generator to create Step 2.



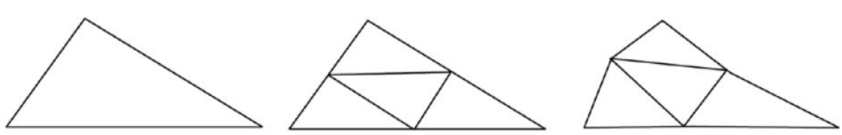
Step 1 Step 2

We can repeat this process to create later steps. Repeating this process can create intricate tree shapes.



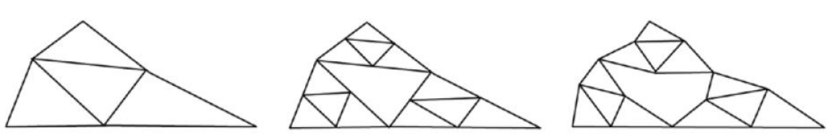
Step 3 Step 4 Final shape

Example 4. Create a variation on the Sierpinski gasket by randomly skewing the corner points each time an iteration is made. Suppose we start with the triangle below. We begin, as before, by removing the middle triangle. We then add in some randomness.



Step 0 Step 1 Step 1 with randomnesss

We then repeat this process.



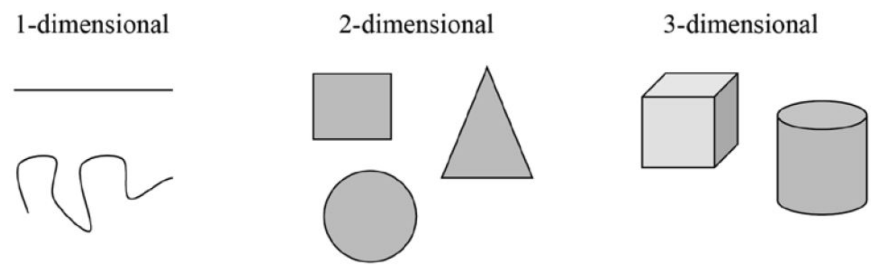
Step 1 with randomnesss Step 2 Step 2 with randomnesss

Continuing this process can create mountain-like structures.

4 Fractal Dimension

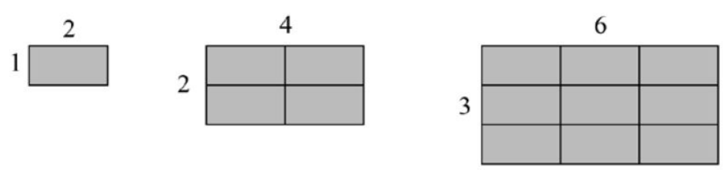
In addition to visual self-similarity, fractals exhibit other interesting properties. For example, notice that each step of the Sierpinski gasket iteration removes one quarter of the remaining area. If this process is continued indefinitely, we would end up essentially removing all the area, meaning we started with a 2-dimensional area, and somehow end up with something less than that, but seemingly more than just a 1-dimensional line.

To explore this idea, we need to discuss dimension. Something like a line is 1-dimensional; it only has length. Any curve is 1-dimensional. Things like boxes and circles are 2-dimensional, since they have length and width, describing an area. Objects like boxes and cylinders have length, width, and height, describing a volume, and are 3-dimensional.

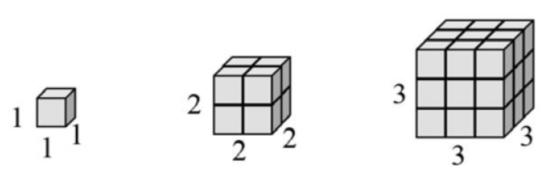


Certain rules apply for scaling objects, related to their dimension. If I had a line with length 1, and wanted scale its length by 2, I would need two copies of the original line. If I had a line of length 1, and wanted to scale its length by 3, I would need three copies of the original.

If I had a rectangle with length 2 and height 1, and wanted to scale its length and width by 2, I would need four copies of the original rectangle. If I wanted to scale the length and width by 3, I would need nine copies of the original rectangle.



If I had a cubical box with sides of length 1, and wanted to scale its length and width by 2, I would need eight copies of the original cube. If I wanted to scale the length and width by 3, I would need 27 copies of the original cube.



Notice that in the 1-dimensional case, copies needed = scale.

In the 2-dimensional case, copies needed = scale^2.

In the 3-dimensional case, copies needed = scale^3.

From these examples, we might infer a pattern.

Scaling-Dimension Relation. To scale a $D-$dimensional shape by a scaling factor S, the number of copies C of the original shape needed will be given by:

Copies = Scale^{Dimension}, or $C = S^D$.

Example 5. Use the scaling-dimension relation to determine the dimension of the Sierpinski gasket.

Suppose we define the original gasket to have side length 1. The larger gasket shown is twice as wide and twice as tall, so has been scaled by a factor of 2. Notice that to construct the larger gasket, 3 copies of the original gasket are needed.

Using the scaling-dimension relation $C = S^D$, we obtain the equation $3 = 2^D$.

Since $2^1 = 2$ and $2^2 = 4$, we can immediately see that D is somewhere between 1 and 2; the gasket is more than a 1-dimensional shape, but we've taken away so much area its now less than 2-dimensional.

Solving the equation $3 = 2^D$ requires logarithms. If you studied logarithms earlier, you may recall how to solve this equation (if not, just skip to the box below and use that formula):

$3 = 2^D$ Take the logarithm of both sides,

$\log(3) = \log(2^D)$ Use the exponent property of logs,

$\log(3) = D\log(2)$ Divide by $\log(2)$,

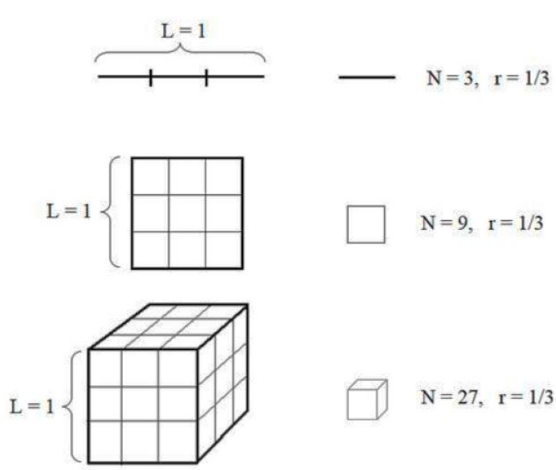
$D = \frac{\log(3)}{\log(2)} \approx 1,585$ The dimension of the gasket is about 1.585.

Scaling-Dimension Relation, to find Dimension to find the dimension D of a fractal, determine the scaling factor S and the number of copies C of the original shape needed, then use the formula:

$$D = \frac{\log(C)}{\log(S)}, \tag{4.1}$$

The first mathematical apparatus for calculating fractional dimensions was proposed by the German scientist Felix Hausdorff in 1919. To obtain a formula for calculating the size of a figure, we can reason as follows. If we take a linear segment and divide it into $N = 3$ equal parts, then the length of each fragment will be three times less than the original length. Let the initial length be conditionally equal to 1, then the lengths of the fragments will be equal to $r = \frac{1}{3}$. Obviously, the total length of the segment is:

$$N \cdot r = 1$$



Let's do the same operation with a square. We'll also divide each of its sides, each one unit long, into three equal parts. That is, the linear dimensions of the small squares will be $r = \frac{1}{3}$. And there are only $N = 9$ such squares. In this case, the area of the large square will be:

$$N \cdot r^2 = 1.$$

As you may have guessed, if you take a cube and also divide each of its sides into three equal segments, you will get $N = 27$ cubes with sides $r = \frac{1}{3}$. Then the volume of the cube can be expressed by the formula:

$$N \cdot r^3 = 1.$$

That is, look, the dimension of the figure is manifested as the degree of the similarity coefficient r. And in the general case, we can write:

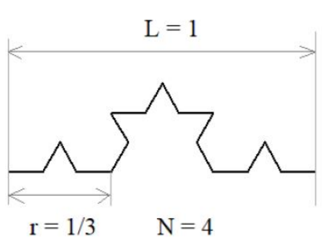
$$N \cdot r^d = 1.$$

To express the power d from this formula, we take the logarithm of the left and right sides of this equation and obtain:

$$\log(N \cdot r^d) = \log 1,$$
$$\log N + d \cdot \log r = 0,$$
$$d \cdot \log r = -\log N,$$
$$-d \cdot \log \frac{1}{r} = -\log N,$$
$$d = \frac{\log N}{\log \frac{1}{r}}.$$

The logarithm can be taken to any base, usually 2, 10 or $e-$the natural logarithm. The value d is called the fractal dimension or similarity dimension.

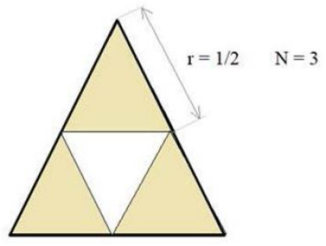
Let's use this formula, calculate the dimension of the Koch curve and check if it is really equal to 1,2618.



It is known that the length of the reduced segments is equal to $r = \frac{1}{3}$ of the total length $L = 1$. There are $N = 4$ such segments in total. Substituting the values into the fractal dimension formula, we have:

$$d = \frac{\log N}{\log \frac{1}{r}} = \frac{\log 4}{\log 3} \approx 1,2618.$$

Let's do the same thing to calculate the dimension of the Sierpinski carpet fractal.

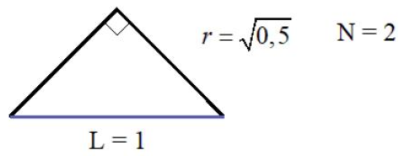


Here the similarity coefficient $r = \frac{1}{2}$. There are $N = 3$ reduced triangles in total. We obtain the dimension of this fractal equal to:

$$d = \frac{\log N}{\log \frac{1}{r}} = \frac{\log 3}{\log 2} \approx 1,58.$$

And also the next example is the dimension of the fractal "Harter-Haytheway dragon".

At each iteration, the straight lines are replaced by rectangular corners. If the length of the straight line is taken as one $(L = 1)$, then the lengths of the sides of the rectangular corner will



be equal to $r = \sqrt{0,5}$. There are $N = 2$ such sides in total. We obtain the dimension of this fractal:

$$d = \frac{\log N}{\log \frac{1}{r}} = \frac{\log 2}{\log \frac{1}{\sqrt{0,5}}} \approx 2.$$

That is, with the number of iterations tending to infinity, the dragon will cover the entire two-dimensional space. In fact, this is one example of a Peano curve, which, being one-dimensional, is capable of going around all points on the plane. And the Hausdorff metric confirms this.

5 Applications of fractals

Fractal shapes exist throughout the human body, in the lungs, blood vessels and neurons. Fractals can also be used to diagnose heart rhythm disorders and tumors. Fractals have a wide range of applications in mathematics and other fields. Some of the most notable applications include:

1. In natural phenomena and geological formations.
2. In medical imaging and biomedical analysis.
3. In computer graphics and visual effects.
4. In financial markets and econophysics.
5. In the field of artificial intelligence and pattern recognition.
6. In art, etc.

Let's learn about them briefly.

Fractals of different dimensions can also model various structures in nature, especially those that have self-similarity across different scales. Consequently, fractal geometry has proven useful in the study of many natural phenomena and geological formations. Coastlines from a distance may look smooth due to slight changes in sea level, but up close we see jagged corners and bays of varying scales - fractal analysis gives us insight into the complex structures of nature.

Fractals are used as a research tool in medical imaging, which includes the use of MRI and computed tomography. Over the past decade, fractal analysis has been applied to much more complex biological structures by studying the branching patterns of neurons, providing vital clues to understanding their structure and function.

Fractals are widely used in computer graphics and visual effects. With their help, the created textures and landscapes of video games, films and virtual environments look extremely realistic. The fractal algorithm is widely used to create realistic natural landscapes such as game terrain, clouds and trees that look like in real life.

We can use fractal geometry to analyze financial markets, using econophysics to model irregular, complex patterns in stock prices and interest rates, as well as the prices of other financial instruments. We can use fractal analysis to predict market trends and evaluate market patterns.

They are used by artificial intelligence and pattern recognition to analyze and classify big data. Fractal-based algorithms can be used in image processing, speech recognition, or machine learning to extract meaningful patterns from noisy data.

Artists use fractal geometry as a basis to create visually arresting works of art. They use mathematical algorithms to generate complex fractal patterns, which are then translated into paintings, digital art, sculptures and other forms of visual expression.

Fractals provide a powerful platform for modeling, analyzing and creating complex patterns and structures in a variety of fields, making them valuable tools in science, technology, engineering, mathematics and the arts.

6 Conclusion

In mathematics, fractals are sets of points in Euclidean space that have a fractional metric dimension or a metric dimension different from the topological one. The term "fractal" was introduced by Benoit Mandelbrot in 1975 and became widely known with the publication of his book "Fractal Geometry of Nature" in 1977. Fractals are widely used in computer graphics to construct images of natural objects such as trees, bushes, mountain landscapes, sea surfaces, and so on.

Today, fractals are widely used in a variety of fields, from mathematics to art:

They are used to describe various phenomena in classical mechanics, hydrodynamics, electrodynamics, and geophysics.

In telecommunications, they allow modeling electromagnetic fields in cellular and satellite communications.

In biology, they accurately describe the structure of natural objects, model and predict their behavior.

Medicine uses fractals to study internal processes in the human body, study heart rhythm, blood vessels, and the nervous system.

In economics, fractals are used to analyze markets and identify patterns in price behavior.

In 3D graphics, they are used to create complex textures and patterns, such as trees, clouds, and sea waves.

In art and design, they are used when you need to create a non-standard "psychedelic" composition, immerse the viewer in new dimensions.

In the future, the authors plan to use fractal calculations for works [13]-[18].

Acknowledgment

The author expresses his sincere gratitude to the reviewers and editors for their useful comments.

References

[1] B.B. Mandelbrot, *Fractals and Multifractals: Noise, Turbulence and Galaxies*, New York, Springer, (1988).

[2] B.B. Mandelbrot, *The Fractal Geometry of Nature*, New York: W.H. Freeman, (1982).

[3] A. Bunde, S. Havlin, *Fractals in Science*, Springer, (2013).

[4] M.F. Barnsley, *Fractals Everywhere*, New York: Academic press, (1988).

[5] E. Gerald, *Measure, Topology, and Fractal Geometry*, 2nd ed. New York, NY: Springer (2008).

[6] F. Kenneth, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed. England: Wiley, (2006).

[7] W. Stephen, *General Topology*, Mineola, NY: Dover, (2004).

[8] D.L. Turcotte, Fractals and fragmentation, *Geophys. Res.*, **91**, 1921–1926 1986.

[9] A.K. Golmankhaneh, *Fractal Calculus and its Applications*, World Scientific, (2022).

[10] A. K. Golmankhaneh, K. Welch, C. Tunç, Y.S. Gasimov, Classical mechanics on fractal curves, *Eur. Phys. J. Special Topics*, **232**, 991–999 (2023).

[11] A.K. Golmankhaneh, K. Welch, C. Serpa, P.E. Jørgensen, Non-standard analysis for fractal calculus, *J. Anal.*, **31**, 1895–1916 (2023).

[12] K. Kamal Ali, A.K. Golmankhaneh, R. Yilmazer, M. Ashqi Abdullah, Solving fractal differential equations via fractal Laplace transforms, *J. Appl. Sci.*, **28:2**, 237–250 (2022).

[13] D.A. Juraev, The Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain, *Siberian Electronic Mathematical Reports*, **14**, 752–764 (2017).

[14] D.A. Juraev, On the Cauchy problem for matrix factorizations of the Helmholtz equation in a bounded domain, *Siberian Electronic Mathematical Reports*, **15**, 11–20 (2018).

[15] D.A. Zhuraev, Cauchy problem for matrix factorizations of the Helmholtz equation, *Ukrainian Mathematical Journal*, **69:10**, 1583–1592 (2018).

[16] D.A. Juraev, On the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain in $\mathbb{R}^2$, *Siberian Electronic Mathematical Reports*, **15**, 1865–1877 (2018).

[17] D.A. Juraev, On a regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation, *Advanced Mathematical Models & Applications*, **4:1**, 86–96 (2019).

[18] D.A. Juraev, The solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation, *Advanced Mathematical Models & Applications*, **5:2**, 205–221 (2020).

Author information

D.A. Juraev, 1. Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, UZBEKISTAN; 2. Department of Mathematics, Anand International College of Engineering, Jaipur, 303012, INDIA.
E-mail: juraevdavron12@gmail.com

N.M. Mammadzada, State Oil Company of the Azerbaijan Republic, Oil and Gas Scientific Research Project Institute, Baku, AZ1122, AZERBAIJAN.
E-mail: memmedzadenazire@gmail.com

Received: 01.04.2024
Accepted: 31.05.2024
Published: 30.06.2024