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# ON SOME COMPARISION OF THE NUMERICAL METHODS APPLIED TO SOLVE ODES, VOLTERRA INTEGRAL AND INTEGRO DIFFERENTIAL EQUATIONS

G.A. Aghayeva, V.R. Ibrahimov and D.A. Juraev

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**Abstract:** The many problems of the different fields of nature are reduce to solve initial-value problem for the both Ordinary Differential Equation and Volterra integro-differential equation and also to solve Volterra integral equation. And for solving theser problems in usually are used the numerical methods, which are related with the development of computer sciences. In among of them, the multistep multiderivative methods are very developing. Therefor, any result in this area is of interest in always. The ordinary multistep method and multistep second derivative methods fundamentally investigated by Dahlquist. The Dahlquist theory development by Ibrahimov, who have receive similarly results for the advanced multistep methods and advanced multistep second derivative methods. Here by development of these results, have considered to investigation of the multistep third derivative methods with constant coefficients. Somebody can be take the results receiving for the estimation of the accuracy for stable and unstable multistep multiderivative methods with third derivative as the simple generalization of the known results. Here have shown that this is not correct. Noted that multistep multiderivative Methods with third derivative here investigated in fool form, found some connection between degree and order for these methods in the cases: stable and in stable. Constructed some concrete methods of multistep third derivative types and recommended some way for the application of these methods to solve some problems.

**Keywords and phrases:** Multistep method with third derivative, stable and degree, the conseption degree and stability, advanced multistep third derivaties methods, explicit and implicit methods.

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## 1 Introduction

As is known, very often we are faced with solving the initial-value problem for the Ordinary differential Equation, which can be presented as the follows:

$$F(t, x, \delta_1 \dot{x}, \delta_2 \ddot{x}, ..., \delta_r x^{(r)}) = 0, \;\; x^{(j)}(t_0) = x_0^{(j)} \;\; (j = 0, 1, 2, .., r-1). \tag{1.1}$$

The numerical solution of this problem has been well studied for the case $j \leq 2$. For the finding the numerical solution of this problem constructed and investigated some class methods of one and multistep types. Among of them one can be recommending some of classes multistep methods with the constant coefficients. Noted that some specialists usually recommended to reducing of the problem (1.1) to the system of differential equation of the first order.

However, adaptation methods for solving ordinary differential equation of higher orders can be more accurate. By taking into account of this, let us to consider investigation of the numerical solution of the following initial-value problem:

$$y'''(x) = f(x, y(x), \delta_1 y'(x), \delta_2 y''(x)), \;\; y^{(j)}_{(x_0)} = y_0^{(j)} \;\; (j = 0, 1, 2), \;\; x_0 \leq x \leq X. \tag{1.2}$$

For the finding the numerical solution of this problem assume that continues of totality of arguments function by, its $f(t, z)$ is defined in some closed set in some which has the continuous partial derivatives up to $P$, inclusively. To construct numerical methods for solving problem

(1.1), let us denote by $y(x_i)$ the exact value of the solution to problem (1.1) at the mesh-point $x_i \ \ (i = 0, 1, ..., N)$. And by the $y_i$ denote approximately values of the numerical solution of the problem (1.1) at the corresponding mesh-point. Here $x_{i+1} = x_i + h \ \ (i = 1, 2, ..., N - k)$ and $h > 0$.

In usually for the finding numerical solution of this problem are used the following multistep methods with constant coefficients (see for example [3], [15], [29], [34] and [37]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{m} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{t} \gamma_i y''_{n+i} + h^3 \sum_{i=0}^{s} l_i f_{n+i}, \ \ n = 0, 1, ..., . \tag{1.3}$$

It is clear that by choosing the constant $k, m, s, t$ one can be construct the numerical methods with the different properties. In the future, will be assume that the amount of the points one and the same; $k = m = t = s$.

It is not difficult to prove that, this method can be applied with equal success to solve the problems below:

$$y''(x) = \varphi(x, y, \delta_1 y'), \ \ y(x_0) = y_0, \ \ y'(x_0) = y_0, \ \ x_0 \le x \le X, \tag{1.4}$$

$$y'(x) = F(x, y), \ \ y(x_0) = y_0, \ \ x_0 \le x \le X. \tag{1.5}$$

Noted that method (1.3) in the application to solve problem (1.4), usually used for the values $l_i = 0 \ (i = 0, 1, ..., s)$, but for the problem (1.5), usually it's coefficients choosing as the $\gamma_i = 0 \ (i = 0, 1, ..., t)$and $l_i = 0 \ (i = 0, 1, ..., s)$. Noted that method (1.3) is more accurate, than the others. It is known that all the methods has own advantages and disadvantages. Methods receiving from the formula (1.3) is more accurate, but in the application that to solve any problems arises some question about the calculation values $y'_i$ and $y''_i$ therefor the next paragraph is devoted to the study of this issue.

## 2 Determining the maximum value of accuracy of the method (1.3)

It is obvious that one of the main issues in the study of numerical methods is their convergence. It is known that the convergence of the multistep methods determined by using the values of the coefficients, which are usually called as stability of the multistep methods. Therefor let us to consider defined the stability of the Multistep Methods with constant coefficients. The definition of the conception of stability for the method (1.3), can it be formulated in the following form:

**Definition 2.1.** The method (1.3) is called as the stable, if the roots of the following polynomial

$$\rho(\lambda) \equiv \alpha_k \lambda^k + \alpha_{k-1} \lambda^{k-1} + \alpha_1 \lambda + \alpha_0$$

located in the unit circle, on the boundary of which there is not multiple root.

Have been proven, that the conception of stability is the necessary and sufficient for the convergence of the multistep method of type (1.3), if the condition $|\beta_m| + |\beta_{m+1}| + ... + |\beta_1| + |\beta_0| \neq 0$ is holds (see for example [30], [35]). Noted that if the specified condition does not hold, then the method (1.3) does not converge. Consequently, if $\beta_i = 0 \quad (i = 0, 1, ..., m)$, then the structure of method (1.3) is changes.

Let us noted that, in solving some problems arises for the study of methods, when $\beta_i = 0, \quad (i = 0, 1, ..., m)$. For example, let us to consider solving problem (1.3) in the case $\delta_1 = 0$. As is known one of the effective methods for solving the problem (1.4) is the Stermer method, which can be presented as the following form (see for example [16]-[23] and [39]-[40]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}, \ \ n = 0, 1, 2, ..., . \tag{2.1}$$

The concept of the stability for the method (2.1) can be formulated in the following form:

**Definition 2.2.** Method (2.1) is called as the stable, if the roots of the polynomial

$$\rho(\lambda) = \alpha_k \lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_1 \lambda + \alpha_0$$

are located in the unite circle on the boundary of which there is not multiple roots except for double root $\lambda = 1$ . Noted that the maximal value for the degree of the method (2.1) and of the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i}, \; n = 0, 1, 2, ..., \quad (2.2)$$

are the same. This method investigated by many authors (see for example [5]-[13], [40]-[41] and [44]).

By simple comparison methods (2.1) and (2.2) receive that these class of methods are separate class methods. And by the comparison of the classes (1.3) and (2.1), receive that the classes (1.3) and (2.1) are also the separate. Therefor these class of methods are investigated in separate form. Noted that for the construction more exact methods one can used corresponding integral equations or other ways (see for example [24]-[27], [30]-[33], [42], [45] and [54]).

And now let us to consider investigation of numerical problem (1.2). For this aim let us to use the formula (1.3). It is easy to understand that for the application of the method (1.3) it is necessary to use methods for calculation of the value $y'_m$ and $y''_m$. In this case arises some difficult related with the calculation of the values $y'_m$ and $y''_m$ with the corresponding to accuracy. Noted that the conception of order of accuracy for the Multistep Methods usually is as the degree, which can be determined as the following.

**Definition 2.3.** The integer values is called as the degree for the method (1.3) (in the case: $(s = t = m = k)$, if the following is holds:

$$\sum_{i=0}^{k} (\alpha_i y(x+ih) - h\beta y'(x+ih) - h^2\gamma_i y''(x+ih) - h^3 l_i y'''(x+ih)) = O(h)^{p+1}, \; h \to 0. \quad (2.3)$$

As is known, one of the basic question in the application of the method (1.3) is the defined the value of $p_{\max}$(maximum value for $p$). For this aim, let us to consider the following expansions, which are satisfies for smooth function:

$$y(x+ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2!} y''(x) + ... + \frac{(ih)^p}{p!} y^{(p)}_{(x)} + O(h)^{p+1}, \quad h \to 0. \quad (2.4)$$

$$y'(x+ih) = y'(x) + ihy''(x) + \frac{(ih)^2}{2!} y'''(x) + ... + \frac{(ih)^{p-1}}{(p-1)!} y^{(p-1)}_{(x)} + O(h)^p, \quad h \to 0. \quad (2.5)$$

$$y''(x+ih) = y''(x) + ihy'''(x) + \frac{(ih)^2}{2!} y^{IV}_{(x)} + ... + \frac{(ih)^{p-2}}{(p-2)!} y^{(p-2)}_{(x)}, \; h \to 0. \quad (2.6)$$

$$y'''(x+ih) = y'''(x) + ihy^{IV}(x) + \frac{(ih)^2}{2!} y^{V}_{(x)} + ... + \frac{(ih)^{p-3}}{(p-3)!} y^{(p-3)}_{(x)} + O(h)^{p-2}, \quad h \to 0. \quad (2.7)$$

Let us suppose that the method (1.3) has the degree of $p$. It follows that the asymptotic equality of (2.3) takes place. Then by using the asymptotic equality (2.4)-(2.7) in the equality of (2.3), receive:

$$\begin{aligned} &\textstyle\sum_{i=0}^{k} \alpha_i y(x) - h \sum_{i=0}^{k} (i\alpha_i - \beta_i) y'(x) - h^2 \sum_{i=0}^{k} (\frac{i^2}{2!}\alpha_i - i\beta_i - \gamma_i) y''(x) - \\ &-h^3 \textstyle\sum_{i=0}^{k} (\frac{i^3}{3!}\alpha_i - \frac{i^2}{2!}\beta_i - i\gamma_i - l_i) y'''(x) - ... - \quad (2.8) \\ &-h^p \textstyle\sum_{i=0}^{k} (\frac{i^p}{p!}\alpha_i - \frac{i^{p-1}}{(p-1)!}\beta_i - \frac{i^{p-2}}{(p-2)!}\gamma_i - \frac{i^{p-3}}{(p-3)!} l_i) y^{(p)}_{(x)} = O(h^{p+1}), \; h \to 0. \end{aligned}$$

By taking into account that the system $1, x, x^2, ..., x^p$ or the system of functions $y(x), y'(x), .., y^{(p)}_{(x)}$ $(y^{(j)}_{(x)} \not\equiv 0,\ j = 0, 1, ..., p)$ is linear independent, one can be write:

$$\sum_{i=0}^{k}\alpha_i = 0;\ \sum_{i=0}^{k}\beta_i = \sum_{i=0}^{k} i\alpha_i;\ \sum_{i=0}^{k}(i\beta_i + \gamma_i) = \sum_{i=0}^{k}\frac{i^2}{2!}\alpha_i; ...;$$
$$\sum_{i=0}^{k}\left(\frac{i^{p-1}}{(p-1)!}\beta_i + \frac{i^{p-2}}{(p-2)!}\gamma_i + \frac{i^{p-3}}{(p-3)!}l_i\right) = \sum_{i=0}^{k}\frac{i^p}{p!}\alpha_i. \tag{2.9}$$

The system of (2.9) is the homogenies linear algebraic equation. The amount of the equation in this system equal $p+1$, but amount of the unknowns equal to $4k+k$. It is known that the homogenies system always has a trivial (zero) solution. Noted that in the application of the method (1.3), suppose that $\alpha_k \neq 0$, because the value$y_{n+k}$ is the solution of the equation (1.3) (usually for the application of the method (3) suppose that the initial values $y_0, y_1, .., y_{k-1}$are known and therefor $y_{n+k}$is the solution of the finite-difference equation (1.3). Without breaking the generality, one can put $\alpha_k = 1$. By taking into the value $\alpha_k = 1$in the system (2.9) receive nonhomogeneous linear algebraic equation. It can be proven that the system (2.9) has the solution different from zero, if $p + 1 \leq 4k + 3$ or $p \leq 4k + 2$. One can be prove that if $p = 4k + 2$, then the corresponding method will be unique and if $p < 4k+2$, then the amount of the corresponding method will be more than 1 (one). Both theoretically and practical interest represent a stable methods so the question arises, about how one can determine maximum value of the degree for the stable methods. For this aim let us to consider the following theorems.

**Theorem 2.4.** *Suppose that the method (1.3) stable and has degree of* $p$ *and* $\alpha_k \neq 0$*, for the case* $(s = t = m = k)$. *Then in the class of the methods (2.8), there are methods with the degree* $p \leq 3k + 4$ *and there are stable method with degree* $p = 3k + 4$ *for* $k = 2j$. *If* $k$*-is odd* $(k = 2j - 1)$, *then there are stable method with the degree* $p = 3k + 3$.

**Theorem 2.5.** *Let us suppose that the method (1.3) has the degree* $p$*, stable,* $\alpha_k \neq 0$ *and* $k \geq \max(m, s, t)$. *Then* $p \leq m + t + s + 4$ *and there are methods with degree* $p = m + t + s + 4$ *for the case* $t + s = -1$ *and* $m = k = 2j$ *also for the* $m = t = s = k = 2j$.

*In the case*

1)$k = 2j,\quad m = 2\nu + 1,\quad t = 2j,\quad s = 2j + 1;$
2)$k = 2m + 1,\quad t = 2j + 1;\quad s = 2\nu,\quad m = 2j;$
3)$k = 2j,\quad m = 2j,\quad t = 2j + 1,\quad s = 2\nu;$
4)$k = 2j + 1,\quad m = 2\nu + 1,\quad t = 2j,\quad s = 2i + 1$

*there are method with degree* $p \leq m + t + s + 2$, *but in other case there are methods with the degree* $p \leq m + t + s + 3$.

Noted that for the construction of more exact stable methods, specialists are constructed hybrid methods (see for example [14], [24]-[27], [38]-[39] and [46]-[55]).

As is known, the method (1.3) is called as the explicit, if $\alpha_k \neq 0$, $\max(m, s, t) < k$ and if $\alpha_k \neq 0$, $\max(m, s, t) = k$, then method (1.3) called as the implicit. If $\alpha_k \neq 0$ and $\max(m, s, t) > k$, then method (1.3) called as the advanced or forward-jumping. Thus have determine the status of receiving methods from the method (1.3) as a special case. And now let us to consider the construction some specific methods.

## 3 About some specific methods with maximum accuracy

Noted that from the method (1.3) one can receive the following explicit and implicit methods the special case:

$$y_{n+2} = y_n + \frac{h(y'_{n+1} + y'_{n+2})}{2} + \frac{h^2(y''_{n+1} - y''_{n+2})}{10} + \frac{h^3(y'''_{n+1} + y'''_{n+2})}{120}, \tag{3.1}$$

$$y_{n+2} = \frac{h(15y'_{n+1} - 13y'_n)}{2} - \frac{h^2(31y''_{n+1} + 29y''_n)}{10} + \frac{h^3(111y'''_{n+1} - 49y'''_n)}{120}, \tag{3.2}$$

$$y_{n+2} = \frac{(y_{n+1} + y_n)}{2} + \frac{h(31y'_{n+1} - 25y'_n)}{4} - \frac{h^2(63y''_{n+1} + 57y''_n)}{20} + \frac{h^3(233y'''_{n+1} - 97y'''_n)}{240}. \tag{3.3}$$

All the methods are stable, method (3.1) is the implicit one step method and has the degree $p = 6$, but methods (3.2) and (3.3) are explicit two step methods and has the degree $p = 6$.

If method (1.3) is stable and has the maximum degree, then the signs of the senior members alternate, that is $\beta_k\gamma_k < 0$ and $\gamma_k l_k < 0$. If $\gamma_k = 0$, then the $\beta_k\gamma_k < 0$is hold. The specified property is mainly used when constructing bilateral methods.

Methods (3.2) and (3.3) are the stable and explicit therefor these methods can be used as the predictor methods, if their accuracy are satisfactory, then methods (3.1) and (3.2) can be used as the main method for solving the given problem. However, often the explicitly methods are used as the predictor formula in the predictor-corrector methods. How does it follow from here, by using methods (3.1)-(3.3) one can construct two different predictor-corrector methods. Noted that, methods (3.1)-(3.3) can be applied to solve some problems, if are known the values of the solution of investigated problem at the points of $x_n$ and $x_{n+1}$.

It is not difficult to understand that methods (3.1)-(3.3), can be applied to solve the initial-value problem for ODEs of the first and second order with the some success. For example let us to consider the following problem:

$$y''(x) = \varphi(x, y, y'), \;\; y(x_0) = y_0, \;\; y'(x_0) = y'(x_0). \tag{3.4}$$

The method (3.2) for the application to solve problem (3.4) can be presented as the following form:

$$y_{n+2} = y_{n+1} + \frac{h(15y'_{n+1} - 13y'_n)}{2} - \frac{h^2(31\varphi_{n+1} + 29\varphi_n)}{10} + \frac{h^3(119g_{n+1} - 49g_n)}{120},$$

here $g(x, y, y') = d\varphi(x, y, y')/dx$.

## 4 Conclusion

As is known, among of the Numerical Methods, which have applied to solve initial-value problems for ODEs, more popular as the Multistep Multiderivative Methods with the constant coefficients. Noted that Multistep Methods and Multistep Second Derivative Methods fundamentally investigated by Dahlquist. But Multistep Multiderivative Methods in some general cases were investigated by Ibrahimov. Here have considering some simple form of the initial-value problem for ODEs of third order. For this have used some concrete methods with the second or third derivative, for the application of which to solve some initial-value problem, one can have constructed predictor-corrector methods in different forms. It is known that in the construction of the predictor-corrector methods are arises some difficulties arise related with properties of the predictor-corrector methods. For example, in some cases as a predictor method one can use the unstable methods. And in some cases one can use the corrector method as the predictor. Note that the receiving results by these predictor-corrector methods are not the same. It is obvious that if the order of the ODEs increases, then the number of methods is also increasing. Therefore, the options for selecting methods are also increasing. And are tried here, describing some ways for a suitable method to obtain better results. Considering the difficulty of constructing algorithms for solving some problems, we tried to construct explicit methods. We hope that the results receiving here will be found over its following.

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## Author information

G.A. Aghayeva, Department of Computational Mathematics, Baku State University, Baku AZ1048, AZERBAIJAN.
E-mail: gulsumm_agayeva@mail.ru

V.R. Ibrahimov, Department of Computational Mathematics, Baku State University, Baku AZ1048, AZERBAIJAN.
E-mail: ibvag47@mail.ru

D.A. Juraev, 1. Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, UZBEKISTAN; 2. Department of Mathematics, Anand International College of Engineering, Near Kanota, Agra Road, Jaipur-303012, Rajasthan, INDIA.
E-mail: juraevdavron12@gmail.com

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