Karshi Multidisciplinary International Scientific Journal

Vol. 1(1) (2024), 49–58

STATISTICAL MODELING OF MULTIVARIATE RELIABILITY FUNCTION USING COPULA METHOD IN THE PRESENCE OF CENSORING

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Communicated by Irfan Nurhidayat

Abstract: Numerous scientific and practical researches conducted around the world are mostly focused use on the methods of statistical modeling. Statistical models play an important role in various fields. Examples include mathematical models in the physics, astronomy, biology, medicine, economics, demography, sociology, psychology, marketing, political science, engineering sciences, machine learning, computer science, natural sciences and other related fields. In this article, we investigate the problem of statistical modeling of multivariate reliability functions. In this case, an algorithm for constructing a statistical model is developed and the copula method is used in the presence of censoring.

Keywords and phrases: Statistical modeling, censoring, copula method, reliability function, dependent variables.

MSC 2010 Classifications: Primary 62G05; Secondary 62G20.

1 Introduction

Statistical modeling is the use of mathematical models and statistical assumptions to generate sample data and make predictions about the real world. A statistical model is a collection of probability distributions on a set of all possible outcomes of an experiment. The nonparametric estimation of a multivariate reliability function will not usually be the end-point of a statistical analysis, but it can play a crucial role in model building and testing on the way to a definitive analysis. Data analysis goals may include estimation of the failure time distribution, and study of dependencies of failure times on study subject characteristics, exposures, or treatments, generically referred to as covariates. Failure time methods have application in many subject matter and research areas, including biomedical, behavioral, physical, and engineering sciences, and various industrial settings (see, [1]).

The joint distribution of a set of random variables $X_1, X_2, ..., X_n$ is defined as

$$F(x_1, x_2, ..., x_n) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n),$$

and the reliability function corresponding to $F(x_1, x_2, ..., x_n)$ is given by

$$R(x_1, x_2, ..., x_n) = P(X_1 > x_1, X_2 > x_2, ..., X_n > x_n).$$

The reliability function is fundamental to a reliability theory, because obtaining reliability probabilities for different values of x provides crucial summary information from reliability data. For $n = 1$,

$$R(x) = 1 - F(x),$$

for $n = 2$

$$R(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + F_1(x_1) \cdot F_2(x_2).$$

Example. For the exponential distribution, which is commonly used to model the time until an event occurs in a process with a constant failure rate, the reliability function can be defined as (see, figure 1):

$$R(x) = e^{-\lambda x},$$

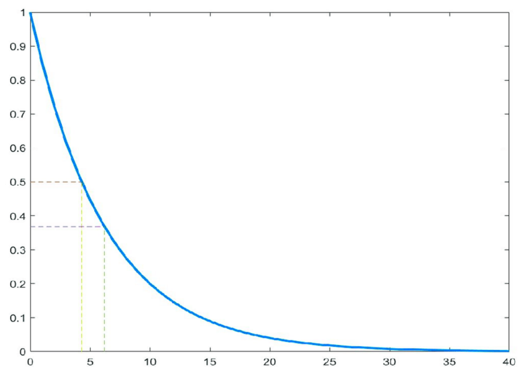

Figure 1. Visualization of the reliability function for the exponential distribution.

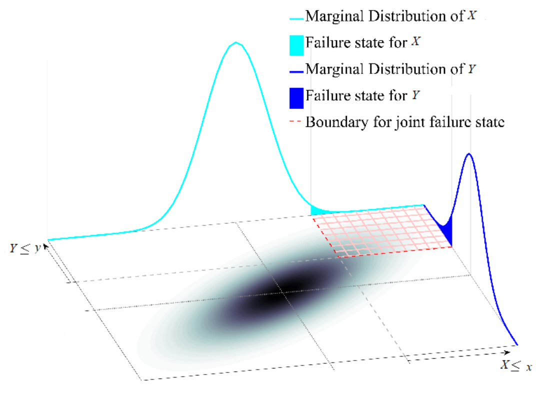

Figure 2. Visualization of copula method.

λ-the item failure rate under the specified operating conditions of temperature, stress, environment, etc.. This formula indicates that the reliability decreases exponentially with time x. As time increases, the probability of the system surviving beyond that time decreases. Therefore, the reliability function is simply the complement of the cumulative distribution function $F(x) = 1 - e^{-\lambda x}$, representing the probability that the event occurs after time x.

2 Copula method in statistical modeling

Learning and using copulas in statistical modeling is a fairly modern method (see, figure 2). By definition, copulas are parametric multivariate distributions generated from uniformly distributed marginals on a given interval $[0, 1]$ (see, [2]). Therefore, the properties of copulas are analogous to the properties of joint distributions.

It is known that the multivariate Poisson distribution, exponential distribution or normal distribution, can be considered as a good model for estimating the joint distribution of abounding microeconomic, macroeconomic and financial variables. But sometimes these distributions may not give good results. This leads to the problem of finding multivariate models that best fit this distribution. One possible way to solve this problem is to use the theory of functions known as copulas. The word copula comes from the Latin for "link" or "tie" together, where the term is used in linguistics to describe such linking words or phrases. The scientific work of mathematicians Hoeffding (1940) and Sklar (1959) played an important role in the initial formation of the theory of these functions (see, [2]-[11]). By the end of the 1990s, the attention of specialists was attracted by mathematical modeling using copula functions and their practical application. The copula function method consists of forming a correlation structure by connecting the marginal distributions to the joint distribution using another auxiliary link function. The use of this method is one of the current problems. The copula methodology separates marginal distributions from

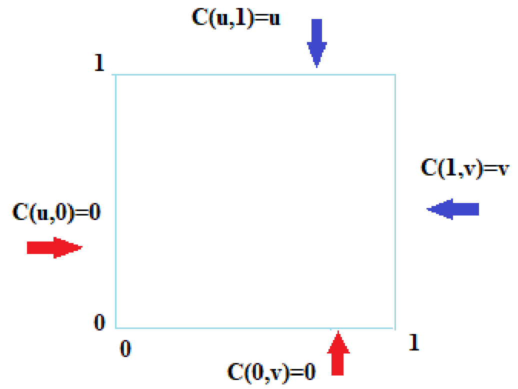

Figure 3. Visualization of limitation property of bivariate copula.

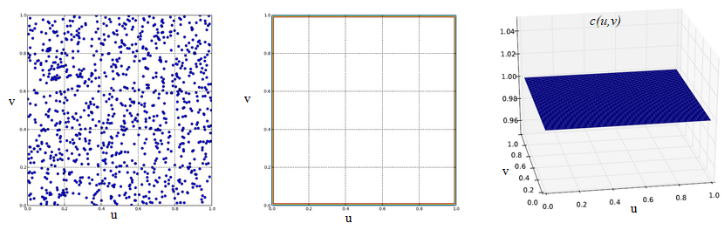

Figure 4. Visualization of Independent copula: (a) scatterplot, (b) contour plot and (c) 3D-view of the copula density.

the dependence structure, linking them to the joint distribution through the copula function. Early monographs on copulas are Joe (1997) (with focus on novel probabilistic notions around copulas) and Nelsen (1999, 2006) (a well-known, readable introduction). An interesting historical perspective and introduction can be found in Durante and Sempi (2010). A more advanced probabilistic treatment of copulas is the recent Durante and Sempi (2015). Now a days, copula functions are successfully used in finance and insurance, biostatistics (Lambert, Vandenhende, 2002)), hydrology (Zhang, Singh, 2006)) and climatology (Salvadori, De Michele, 2007)(see for detail [29]-[33]).

Let us give the definition of the two-dimensional copula function in order to understand its meaning in more detail.

Definition 2.1. (Axiomatical) A two-dimensional copula C is a mapping from $I^2 = [0,1] \times [0,1]$ to $I = [0,1]$ which satisfies the following three conditions:
1.(Limitation) $C(u,0) = C(0,u) = 0$ for every $u, v \in [0,1]$;
2. $C(u,1) = u$, $C(1,v) = v$ for every $u, v \in [0,1]$ (see, figure 3);
3. (Monotonicity) $C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$ for every $u_1, v_1, u_2, v_2 \in [0,1]$ satisfying $u_1 \leq u_2, v_1 \leq v_2$.

In this definition, the first condition is a requirement for any distribution function. The second condition requires uniformity of one-dimensional marginal distributions on $[0,1]$, which is the main idea of copulas. The last property is called the rectangular inequality.

Statistically, a copula can be defined as follows.

Definition 2.2. (Statistical) A copula $C(u,v) : [0,1]^2 \to [0,1]$ is a bivariate distribution function with uniform marginals.

A first example of copulas is the product copula $C(u;v) = uv$ (see, figure 4).

Let X and Y be continuous random variables(r.v.-s) with d.f.-s $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$, and joint distribution function (d.f.) $H(x,y) = P(X \leq x, Y \leq y)$. The importance of copulas in statistics is described in Sklar's Theorem (see, [34]).

Theorem 2.3. *(Abe Sklar, 1959) Let H be a joint d.f. with margins F and G. Then there exists a copula C such that for all x, y in R,*

$$H(x,y) = C(F(x), G(y)). \tag{2.1}$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $Ran(F) \times Ran(G)$. Conversely, if C is a copula and F and G are d.f.-s, then the function H defined by (2.1) is a joint d.f. with margins F and G.

The representation (2.1) suggests that if the copula C were known, then substituting continuous marginal estimators for F and G would yield a plug-in estimate of their associated joint d.f. H.

Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$.

Definition 2.4. Copulas of the form

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)), \tag{2.2}$$

are called Archimedean copulas, where the function φ is called a generator of the copula $\varphi(1) = 0$, $\varphi^{[-1]}$ is the pseudo-inverse function of φ. If $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{-1}$.

Note that $\varphi^{[-1]}$ is continuous and non-increasing on $[0,\infty]$, and strictly decreasing on $[0,\varphi(0)]$. For example, $\varphi(t) = -lnt$ generator of Gumbel-Hougaard copula.

3 Description of the censoring model with covariates

The problem of estimating multivariate distributions based on censored data began in the early 1980s. For example, the scientific research works of the following scientists can be cited: Gregory Campbell (1981), Gregory Campbell & Antonia Földes (1982), James A. Hanley & Milton N. Parnes (1983), Lajos Horváth (1983), Wei-Yann Tsai, Sue Leurgans & John Crowley (1986), Murray D. Burke (1988), Dorota M. Dabrowska (1988, 1989), Richard D. Gill (1992), Abdurakhim Abdushukurov (2004), Abdurakhim Abdushukurov & Rustamjon Muradov (2011, 2013, 2022-2024) and others (see, [5]-[7], [18]-[28], [31], [32]). In particular, if the distribution is two-dimensional, the time until the death of people (twins or couples), the failure times of the components of a physical system can be given as random censored variables.

At present time, there are several approaches to estimating of reliability functions of vectors of lifetimes. Moreover, the random variables (r.v.-s) of interest lifetimes and censoring r.v.-s can be also influenced by other variable, often called prognostic factor or covariate. In medicine, dose a drug and in engineering some environmental conditions temperature, pressure and others are influenced to the observed variables (see, [10], [15], [17]).

Let's consider the case when the support of covariate C is the interval $[0,1]$ and we consider our results on fixed design points $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1$. Thus, on a probability space $(\Omega, \mathcal{A}, P)$. Let's consider two sequences $\mathbb{X} = \{(X_{1i}, X_{2i}), i \ge 1\}$ and $\mathbb{Y} = \{(Y_{1i}, Y_{2i}), i \ge 1\}$ of conditionally independent random vectors with conditional reliability functions $F_x(t,s) = P(X_{11} > t, X_{21} > s/C = x)$ and $G_x(t,s) = P(Y_{11} > t, Y_{21} > s/C = x)$ at given covariate $C = x$, where $x \in [0,1]$ and $(t,s) \in \bar{R}^{+2} = [0,\infty) \times [0,\infty)$. Here sequences $\mathbb{X}$ and $\mathbb{Y}$ can be dependent. Introduce a marginal conditional reliability functions

$$R_{1x}^{X}(t) = P(X_{11} > t/C = x), R_{2x}^{X}(s) = P(X_{21} > s/C = x),$$

$$R_{1x}^{Y}(t) = P(Y_{11} > t/C = x), R_{2x}^{Y}(s) = P(Y_{21} > s/C = x), \tag{3.1}$$

which supposed to be continuous, $x \in [0,1]$, $t, s \ge 0$. Suppose that vector $\mathbb{X}$ is censored from the right by vector $\mathbb{Y}$ and observed data is consist on $\mathbb{V}^{(n)} = \{(Z_i, \delta_i, C_i), i = 1, \ldots, n\}$, where $Z_i = (Z_{1i}, Z_{2i})$, $Z_{ki} = \min(X_{ki}, Y_{ki})$, $\delta_i = (\delta_{1i}, \delta_{2i})$, $\delta_{ki} = I(Z_{ki} = X_{ki})$, $k = 1, 2$;

$i=\overline{1,\,n}$. Here $I\left(A\right)$ is stands for an indicator of event A. Introduce jointly conditional reliability function of the vector $\left(X_{11},X_{21,}Y_{11},Y_{21}\right)$ given $C=x$:

$$K_x\left(t,s,z,v\right)=P\left(X_{11}>t,X_{21}>s,Y_{11}>z,Y_{21}>v/C=x\right),\ \left(t,s,z,v\right)\in\overline{R}^{+4}.$$

Then we obtain representations at $x\in\left[0,1\right]$:

$$F_x\left(t,s\right)=K_x\left(t,s,0,0\right),\quad G_x\left(t,s\right)=K_x\left(0,0,t,s\right),\ \left(t,s\right)\in\overline{R}^{+2}, \tag{3.2}$$

$$H_x\left(t,s\right)=P\left(Z_{11}>t,Z_{21}>s/C=x\right)=P\left(X_{11}>t,Y_{11}>t,X_{21}>s,Y_{21}>s/C=x\right)=$$
$$=K_x\left(t,s,\,t,s\right),\,\left(t,s\right)\in\overline{R}^{+2}. \tag{3.3}$$

Then according to the Theorem of Sklar (see, [2],[34]), there exist on $\left[0,1\right]^4$ conditional copula survival function $C_x\left(\overline{u}\right),\ \overline{u}=\left(u_1,u_2,u_3,u_4\right)\in\left[0,1\right]^4$ such that we have following representation of reliability function (3.2) for all $\left(t,s,z,v\right)\in\overline{R}^{+4}$ as

$$K_x\left(t,s,z,v\right)=C_x\left(R_{1x}^X\left(t\right),R_{2x}^X\left(s\right),R_{1x}^Y\left(z\right),R_{2x}^Y\left(v\right)\right). \tag{3.4}$$

Assume that at the fixed design value $x\in\left[0,1\right]$, C_x is Archimedean copula, i.e.

$$C_x\left(u_1,u_2,u_3,u_4\right)=\varphi^{[-1]}\left[\varphi_x\left(u_1\right)+\varphi_x\left(u_2\right)+\varphi_x\left(u_3\right)+\varphi_x\left(u_4\right)\right],\bar{u}=\left(u_1,u_2,u_3,u_4\right)\in\left[0,1\right]^4, \tag{3.5}$$

where, for each x, $\varphi_x:\left[0,1\right]\to\left[0,\infty\right]$ is a known continuous, convex, strictly decreasing function with $\varphi_x\left(1\right)=0$, $\varphi_x^{[-1]}$ is a pseudo-inverse of φ_x (see, [2], [13]) and given by

$$\varphi_x^{[-1]}(s)=\begin{cases}\varphi_x^{-1}(s), & 0\le s\le\varphi_x(0),\\ 0, & \varphi_x(0)\le s\le\infty.\end{cases}$$

We assume that copula generator function φ_x is strict, i.e. $\varphi_x\left(0\right)=\infty$ and hence $\varphi_x^{[-1]}=\varphi_x^{-1}$. Thus from formulas (3.2)-(3.5) one can get for a fixed $x\in\left[0,1\right]$

$$F_x\left(t,s\right)=\varphi_x^{-1}\left[\varphi_x\left(R_{1x}^X\left(t\right)\right)+\varphi_x\left(R_{2x}^X\left(s\right)\right)\right],\ \left(t,s\right)\in\overline{R}^{+2},$$
$$G_x\left(t,s\right)=\varphi_x^{-1}\left[\varphi_x\left(R_{1x}^Y\left(t\right)\right)+\varphi_x\left(R_{2x}^Y\left(s\right)\right)\right],\ \left(t,s\right)\in\overline{R}^{+2}, \tag{3.6}$$

and

$$H_x\left(t,s\right)=\varphi_x^{-1}\left[\varphi_x\left(R_{1x}^X\left(t\right)\right)+\varphi_x\left(R_{2x}^X\left(s\right)\right)+\varphi_x\left(R_{1x}^Y\left(t\right)\right)+\varphi_x\left(R_{2x}^Y\left(s\right)\right)\right],\ \left(t,s\right)\in\overline{R}^{+2},$$

Since

$$R_{1x}^Z\left(t\right)=P\left(Z_{11}>t/C=x\right)=H_x\left(t;0\right)=P\left(X_{11}>t,\ Y_{11}>t/C=x\right),\ t\ge0,$$

and

$$R_{2x}^Z\left(s\right)=P\left(Z_{21}>s/C=x\right)=H_x\left(0,s\right)=P\left(Y_{11}>s,\ Y_{21}>s/C=x\right),\ s\ge0,$$

then from last formula in (3.2) we obtain

$$R_{1x}^Z\left(t\right)=\varphi_x^{-1}\left[\varphi_x\left(R_{1x}^X\left(t\right)\right)+\varphi_x\left(R_{1x}^Y\left(t\right)\right)\right],\ t\ge0,$$
$$R_{2x}^Z\left(s\right)=\varphi_x^{-1}\left[\varphi_x\left(R_{2x}^X\left(s\right)\right)+\varphi_x\left(R_{2x}^Y\left(s\right)\right)\right],\ s\ge0. \tag{3.7}$$

Now from (3.7) it follows that for fixed $x\in\left[0,1\right]$

$$\varphi_x\left(H_x\left(t,s\right)\right)=\varphi_x\left(F_x\left(t,s\right)\right)+\varphi_x\left(G_x\left(t,s\right)\right),\ \left(t,s\right)\in\overline{R}^{+2}.$$

Note that functionals (3.7) admits to estimation of one dimensional reliability functions R_{1x}^X and R_{2x}^X correspondingly by subsamples

$$\mathbb{V}_1^{(n)}=\left\{\left(Z_{1i},\delta_{1i},C_i\right),\,i=\overline{1,n}\right\}$$

and

$$\mathbb{V}_2^{(n)} = \left\{ (Z_{2i}, \delta_{2i}, C_i) \, , \, i = \overline{1, n} \right\}$$

with $\mathbb{V}_1^{(n)} + \mathbb{V}_2^{(n)} = \mathbb{V}^{(n)}$ and then by $\mathbb{V}^{(n)}$ estimation of joint reliability function $F_x(t,s)$ using first formula in (3.2). Let $H_{kx}^{(1)}(t) = P(Z_{ki} \le t, \delta_{ki} = 1/C_i = x)$ are subdistribution functions and $\Lambda_{kx}(t)$ is crude hazard functions of r.v. X_{ki}, $k = 1, 2$ subjecting to censoring by Y_{ki} for given $C_i = x$. Then (see, [1]) we have

$$\Lambda_{kx}(dt) = \frac{P(X_{ki} \in dt, \; X_{ki} \le Y_{ki}/C_i = x)}{P(X_{ki} \ge t, \quad Y_{ki} \ge t/C_i = x)} = \frac{H_{kx}^{(1)}(dt)}{R_{kx}^Z(t-)}, \;\; k = 1, 2. \tag{3.8}$$

From (3.4) one can obtain following expressions of reliability functions R_{kx}^X :

$$R_{kx}^X(t) = \varphi_x^{-1}\left\{ -\int_0^t R_{kx}^Z(u-)\,\varphi_x'\left(R_{kx}^Z(u)\right) d\Lambda_{kx}(u) \right\} =$$

$$= \varphi_x^{-1}\left\{ -\int_0^t \varphi_x'(R_{kx}^Z(u))dH_{kx}^{(1)}(u) \right\}, \; t \ge 0, \; k = 1, 2, \tag{3.9}$$

(see, for example, [17]). In order to constructing the estimator of S_{kx}^X according to representation (3.9), we introduce some smoothed estimators of R_{kx}^Z and $H_{kx}^{(1)}$. Similarly to Breakers and Veraverbeke (2005), we will also use the Gasser-Müller weights

$$w_{ni}(x, h_n) = \frac{1}{q_n(x, h_n)} \int_{x_{i-1}}^{x_i} \frac{1}{h_n} \pi\left(\frac{x-u}{h_n}\right) du, \;\; i = 1, ..., n,$$

with

$$q_n(x, h_n) = \int_0^{x_n} \frac{1}{h_n} \pi\left(\frac{x-u}{h_n}\right) du,$$

where $x_0 = 0$, π is a known probability density function (kernel) and $\{h_n, \; n \ge 1\}$ is a sequence of positive constants, tending to zero as $n \to \infty$, called bandwidth sequence. Let's introduce the weighted estimators of R_{kx}^Z and $H_{kx}^{(1)}$ respectively for $k = 1, 2$ as

$$R_{kx}^Z(t) = \sum_{i=1}^n w_{ni}(x, h_n)\, I(Z_{ki} > t) = 1 - H_{kxh}(t),$$

$$H_{kxh}(t) = \sum_{i=1}^n w_{ni}(x, h_n)\, I(Z_{ki} \le t, \; \delta_{ki} = 1). \tag{3.10}$$

$$R_{kx}^Z(t) = \sum_{i=1}^n w_{ni}(x, h_n)\, I(Z_{ki} > t) = 1 - H_{kxh}(t),$$

$$H_{kxh}(t) = \sum_{i=1}^n w_{ni}(x, h_n)\, I(Z_{ki} \le t, \; \delta_{ki} = 1).$$

$$R_{kxh} = \varphi_x^{-1}\left\{ -\int_0^t \varphi_x'\left(R_{kxh}^Z(u)\right) dH_{kxh}^{(1)}(u) \right\}, \; t \ge 0. \tag{3.11}$$

Remark that in case of no covariate, estimator (3.11) reduces to estimator first obtained by Zeng and Klein (1995)(see, [29]) for one sample case, which in case of the independent copula $\varphi(y) = -\log y$, reduces to a exponential hazard estimate. Now we propose second estimator from [15], the extended analogue of relative-risk power estimator introduced in [4], [5] under independent censoring case for $k=1,2:$

$$\hat{R}^X_{kxh}(t)=\varphi_x^{-1}\left[\varphi_x\left(\hat{R}^Z_{kxh}(t)\right)\mu_{kxh}(t)\right],\ t\geq 0, \tag{3.12}$$

where

$$\mu_{kxh}(t)=\varphi_x\left(R^X_{kxh}(t)\right)\left[\varphi_x\left(\tilde{R}^Z_{kxh}(t)\right)\right]^{-1},$$

$$\varphi_x\left(R^X_{kxh}(t)\right)=-\int\limits_0^t\varphi_x^{'}\left(R^Z_{kxh}(u)\right)dH^{(1)}_{kxh}(u),$$

$$\varphi_x\left(\hat{R}^Z_{kxh}(t)\right)=-\int\limits_0^t n\left[\varphi_x\left(R^Z_{kxh}(u)\right)-\varphi_x\left(R^Z_{kxh}(u)-\frac{1}{n}\right)\right]dH^{(1)}_{kxh}(u),$$

$$\varphi_x\left(\tilde{R}^Z_{kxh}(t)\right)=-\int\limits_0^t\varphi_x^{'}\left(R^Z_{kxh}(u)\right)dH_{kxh}(u).$$

Using estimators (3.11) and (3.12) we can propose two corresponding estimators of the first jointly reliability function in (3.6):

$$F_{xh}(t,s)=\varphi_x^{-1}\left[\varphi_x\left(R^X_{1xh}(t)\right)+\varphi_x\left(R^X_{2xh}(s)\right)\right],\ (t,s)\in\overline{R}^{+2},$$

$$\hat{F}_{xh}(t,s)=\varphi_x^{-1}\left[\varphi_x\left(\hat{R}^X_{1xh}(t)\right)+\varphi_x\left(\hat{R}^X_{2xh}(s)\right)\right],\ (t,s)\in\overline{R}^{+2}. \tag{3.13}$$

In order to investigate estimators (3.13) we introduce some conditions. For the design point $x_1,...,x_n$ denote

$$\underline{\Delta}_n=\min_{1\leq i\leq n}(x_i-x_{i-1}),\ \bar{\Delta}_n=\max_{1\leq i\leq n}(x_i-x_{i-1}).$$

For the kernel π, let $\|\pi\|_2^2=\int\limits_{-\infty}^{\infty}\pi^2(u)\,du$, $m_v(\pi)=\int\limits_{-\infty}^{\infty}u^v\pi(u)\,du,\ v=1,2,$ $\|\pi\|_\infty=\sup\limits_{u\in\mathbb{R}}\pi(u)$.
Moreover, we use next assumptions on design points and on the kernel function:

A1) As $n\to\infty,\ x_n\to 1,\ \underline{\Delta}_n=\mathrm{O}\left(\frac{1}{n}\right),\ \ \bar{\Delta}_n-\underline{\Delta}_n=o\left(\frac{1}{n}\right).$

A2) π is a probability density function with compact support $[-M,\,M]$ for some $M>0$, with $m_1(\pi)=0$ and $|\pi(u)-\pi(u')|\leq C(\pi)|u-u'|$, where $C(\pi)$ is some constant. Let $T_k=\inf\left\{t\geq 0:\ S^Z_{kx}(t)=0\right\},\ k=1,2$. For our result we need some smoothness conditions on functions $H_{kx}(t)=P(Z_{ki}\leq t/C_i=x)$ and $H^{(1)}_{kx}(t),\ k=1,2$. We formulate them for a general (sub) distribution function $N_x(t),\ 0\leq x\leq 1,\ \ t\geq 0$ and for a fixed $T>0:$

A3) $\frac{\partial}{\partial x}N_x(t)=\dot{N}_x(t)$ exist and continuous in $(x,t)\in[0,1]\times[0,T]$.

A4) $\frac{\partial}{\partial t}N_x(t)={N'}_x(t)$ exist and continuous in $(x,t)\in[0,1]\times[0,T]$.

A5) $\frac{\partial^2}{\partial x^2}N_x(t)=\ddot{N}_x(t)$ exist and continuous in $(x,t)\in[0,1]\times[0,T]$.

A6) $\frac{\partial^2}{\partial t^2}N_x(t)={N''}_x(t)$ exist and continuous in $(x,t)\in[0,1]\times[0,T]$.

A7) $\frac{\partial^2}{\partial x\partial t}N_x(t)=\dot{N}'_x(t)$ exist and continuous in $(x,t)\in[0,1]\times[0,T]$.

A8) $\frac{\partial\varphi_x(u)}{\partial u}=\varphi_x^{'}(u)$ and $\frac{\partial^2\varphi_x(u)}{\partial u^2}=\varphi_x^{''}(u)$ are Lipschitz in the $x-$ direction with a bounded Lipschitz constant and $\frac{\partial^3\varphi_x(u)}{\partial u^3}=\varphi_x^{'''}(u)$ exist and continuous in $(x,t)\in[0,1]\times[0,T]$.

4 Asymptotic properties of estimators

Firstly we establish the asymptotic equivalence of two estimators $\hat{F}_{xh}(t_1,t_2)$ and $F_{xh}(t_1,t_2)$ proposed by formulas (3.13). Let's denote $\nabla=[0,T_1]\times[0,T_2]$.

Theorem 4.1. *Assume (A1), (A2), functions $H_{kxh}(t)$ and $H^{(1)}_{kxh}(t)$ satisfy (A5)-(A7) in $[0,T_k],\ k=1,2$; φ_x satisfies (A8). Then as $n\to\infty$ and $x\in[0,1]:$*

$$\sup_{(t_1,t_2)\in\nabla}\left|\hat{F}_{xh}(t_1,t_2)-F_{xh}(t_1,t_2)\right|\overset{a.s.}{=}\mathrm{O}\left(\frac{1}{n}\right). \tag{4.1}$$

Thus for all $(t_1,t_2)\in\nabla$ and $x\in[0,1]$

$$\hat{F}_{xh}(t_1,t_2)-F_x(t_1,t_2)=F_{xh}(t_1,t_2)-F_x(t_1,t_2)+q_n(t_1,t_2),$$

where $q_n(t_1,t_2)=\hat{F}_{xh}(t_1,t_2)-F_{xh}(t_1,t_2)$ and by Theorem 4.1 $\sup\limits_{(t_1,t_2)\in\nabla}|q_n(t_1,t_2)|\overset{a.s.}{=}\mathrm{O}\left(\frac{1}{n}\right).$
Hence from two estimators in (3.13) it is enough to investigate only estimator $F_{xh}(t_1,t_2)$. The next result is on almost sure representation of estimator by sums of weighted functions with rate.

Theorem 4.2. *Assume (A1), (A2), functions $H_{kxh}(t)$ and $H_{kxh}^{(1)}(t)$ satisfy (A5)-(A7) in $[0,T_k]$, $k=1,2$; φ_x satisfies (A8), $h_n\to 0$, $\frac{\log n}{nh_n}\to 0$ and $\frac{nh_n^5}{\log n}=\mathrm{O}(1)$. Then as $n\to\infty$,*

$$F_{xh}(t_1,t_2)-F_x(t_1,t_2)=\sum_{k=1,2}\sum_{i=1}^{n}w_{ni}(x,h_n)\,\psi_{kxt_k}(Z_i,\delta_i)+r_n(t_1,t_2),$$

where for $k=1,2$

$$\psi_{kxt_k}(Z_i,\delta_i)=\frac{(-1)}{\varphi_x'(R_{kx}^X(t_k))}\left[\int_0^{t_k}\varphi_x''\left(R_{kx}^Z(u)\right)\left(I(Z_{ki}\le u)-H_{kx}(u)\right)dH_{kx}^{(1)}(u)-\right.$$
$$-\varphi_x'\left(R_{kx}^Z(t_k)\right)\left(I(Z_{ki}\le t,\ \delta_{ki}=1)-H_{kx}^{(1)}(t_k)\right)-$$
$$\left.-\int_0^{t_k}\varphi_x''\left(R_{kx}^Z(u)\right)\left(I(Z_{ki}\le u,\ \delta_{ki}=1)-H_{kx}^{(1)}(u)\right)dH_{kx}(u)\right],$$

$r_n(t_1,t_2)=r_{1n}(t_1)+r_{2n}(t_2)$ *and*

$$\sup_{0\le t\le T_k}|r_k(t_k)|\overset{a.s.}{=}\mathrm{O}\left(\left(\frac{\log n}{nh_n}\right)^{3/4}\right).$$

5 Conclusion

Following results are obtained in the article: Investigated the importance of Archimedean copula functions in statistical modeling. Studied the characteristic properties of the copula function and a visualization graph of the limiting property of a two-dimensional copula has been constructed. Sklar's theorem was applied to statistical estimation of two-dimensional joint distribution and reliability functions. The asymptotically properties of statistical estimates constructed for reliability functions in the presence of covariates are investigated.

Acknowledgment

In conclusion, the authors expresses his sincere gratitude to the professor of Moscow State University Tashkent Branch, the Department of Applied Mathematics and Informatics, Professor, Abdurakhim Abdushukurov and reviewer for useful advice.

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Received: 02.04.2024
Accepted: 25.05.2024
Published: 30.06.2024