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ON SOME COMPARISON OF ADAM'S METHODS WITH MULTISTEP METHODS AND APPLICATION THEM TO SOLVE INITIAL-VALUE PROBLEM FOR THE ODEs FIRST ORDER

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Abstract: Among in the class of Numerical Methods for solving initial-value problem one of the popular methods is the Adams-Moulton and Adams-Bashforth, which make up the Adams-family. Many experts believe that Multistep methods are obtained from the generalization of Adams methods. Historically it happened that first the methods of Adams appeared. And after the emergence of Adams methods specialists constructed methods that is a special case of the Adams methods. Noted that Adams method intersects with the Runge-Kutta methods at one point, which is called Euler's method. Adams methods and Runge-Kutta methods are the intersects at the multiple points in the application them to calculation of definite integrals. As is known the fourth order Runge-Kutta method, which was constructed by Runge, coincides with Simpson's method in the application them to calculation of the definite integral. Here, have compared Adam's methods with Multistep Methods in the application of them to solve initial-value problem for the Ordinary Differential Equations the first order. By using specific examples it is shown, how one can obtain Adams methods from the Runge-Kutta methods and vice versa.

Keywords and phrases: Initial-value problem, ordinary differential equations (ODEs), Adams-Moulton and Adams-Bashforth method, stability and degree, Simson and trapezoid methods, methods of Runge-Kutta.

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1 Introduction

As is known the investigation of the ODEs beginning from the Newton. Construction the Numerical Methods for solving initial-value problem of the ODEs has been investigated by many known authors as Euler, Kowell, A.N.Krilov, Adams, Bakhvalov, Dahlquist, Runge-Kutta, Butcher, Iserls, Norset, Ibrahimov and etc. have constructed some classes Numerical methods to solve initial-value problem for the ODEs. As is known the first Numerical methods are constructed by using power series, which are called as the Numerical-Analytical method. Euler point out the shortcomings of these methods offered his famous direct Numerical method. This direction was developed by many famous scientists, who have constructed different Numerical Methods for solving above named problem.

Let us to consider the following initial-value problem for the ODEs of the first order, having the following form (see for example [1]-[21]):

$$y'(x) = f(x, y), \quad y(x_0) = y_0, \quad x_0 \leq x \leq X. \tag{1.1}$$

For the investigation, the numerical solutions of this problem, let us suppose that the solution *y(x)* of the problem (1.1) is a continuous function that is defined in a segment, where that has derivatives of some order *p+1,* inclusively. But the continuous to totality of arguments function *f(x,y)* is determine that in some closed set in which has partial derivatives to some of order p, inclusively.

Let us divide the segment $[x_0, X]$ to N equal parts by using a constant step-size h and denote the mesh-points by the $x_{i+1} = x_i + h$, here $h > 0$. The exacts values of the solutions of problem (1.1) at the mesh-points x_i, let us denote by the $y(x_i)$ and the corresponding approximately value

by the $y_i\,(i = 0, 1, ..., N)$. The exact values of the function *f(x,y(x))* at the pointdenote by the and the approximately values by the $f(x_i, y_i)$ or $f_i(i = 0, 1, ..., N)$.
The classical method Runge-Kutta can be presented as follows:

$$\tilde{y}_{n+1} = \tilde{y}_n + h(\tilde{\beta}_1\tilde{k}_1 + \tilde{\beta}_2\tilde{k}_2 + ... + \tilde{\beta}_m\tilde{k}_m), \tag{1.2}$$

here $\tilde{k}_i(i = 1, 2, .., m)$ are defined in the following form:

$$\begin{aligned}
&\tilde{k}_1 = f(x_n, y_n),\\
&\tilde{k}_2 = f(x_n + \tilde{\alpha}_2 h,\, y_n + h\gamma_{2,1}\tilde{k}_1),\\
&\tilde{k}_3 = f(x_n + \tilde{\alpha}_3 h,\, y_n + h\gamma_{3,2}\tilde{k}_2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\tilde{k}_m = f(x_n + \tilde{\alpha}_m h,\, y_n + h(\tilde{\gamma}_{m,1}\tilde{k}_1 + \tilde{\gamma}_{m,2}\tilde{k}_2 + ... + \gamma_{m,m}\tilde{k}_m)).
\end{aligned}$$

This method is explicit therefore, it can be easily applied to solve some practical problems.
By using method (1.2), one can construct a semi-implicit one-step method of Runge-Kutta types, which in one version can be present as follows:

$$y_{n+1} = y_n + h(\bar{\beta}_1\bar{k}_1 + \bar{\beta}_1\bar{k}_2 + ... + \bar{\beta}_m\bar{k}_m). \tag{1.3}$$

The quantities involved here are defined as the following form:

$$\begin{aligned}
&\bar{k}_1 = f(x_n + \alpha_1 h,\, y_n + h\bar{\gamma}_{1,1}\bar{k}_1),\\
&\bar{k}_2 = f(x_n + \alpha_2 h,\, y_n + h\bar{\gamma}_{2,1}\bar{k}_1 + h\bar{\gamma}_{2,2}\bar{k}_2),\\
&\bar{k}_3 = f(x_n + \alpha_3 h,\, y_n + h\bar{\gamma}_{3,1}\bar{k}_1 + h\bar{\gamma}_{3,2}\bar{k}_2 + h\gamma_{3,3}\bar{k}_3),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\bar{k}_m = f(x_n + \alpha_m h,\, y_n + h\bar{\gamma}_{m,1}\bar{k}_1 + h\bar{\gamma}_{m,2}\bar{k}_2 + ... + h\bar{\gamma}_{m,m}\bar{k}_m).
\end{aligned}$$

If methods (1.2) and (1.3) are compared with known methods, then receive that these methods resemble ordinary explicit and implicit methods.

As is known application of the method (1.3) arises some difficult, which are related with the calculation the values $\bar{k}_1, \bar{k}_2, ..., \bar{k}_m$. It is obvious that the values $\bar{k}_j(j = 1, 2, ..., m)$ can be calculated by the above given sequence. For the sake of objectivity, let us noted that in calculations of the $\bar{k}_j(j = 1, 2, ..., m)$ are arises necessity to solve some nonlinear algebraic equations. Now let us consider the construction of the implicit Runge-Kutta methods, which can be written as the following:

$$y_{n+1} = y_n + h(\beta_1 k_1 + \beta_2 k_2 + ... + \beta_m k_m), \tag{1.4}$$

here the unknowns are defined in the following form:

$$k_i = f(x_n + \alpha_i h,\, y_n + h(\gamma_{i,1}k_1 + \gamma_{i,2}k_2 + ... + \gamma_{i,m}k_m)), \quad i = 1, 2, ..., m.$$

The aim of our investigation is compares the methods of type Runge-Kutta and Adams. In usually the Adams method, which is applied to solve problem (1.1), is presented in the following form (see for example [22]-[45]):

$$y_{n+k} = y_{n+k-1} + h\sum_{i=0}^{k}\beta_i f_{n+i}, \quad n = 0, 1, ..., N-k. \tag{1.5}$$

For the value $\beta_k = 0$ from this method, one can receive explicit Adams, or Adams-Moulton method. Obviously, the method (1.5) is implicit or is the type of Adams-Bashforth for the case $\beta_k \neq 0$. By the simple comparison, receive that Adams-Moluton method corresponds to method (1.2), but Adams-Bashforht method corresponds to method (1.3). Simple comparison of Adams methods with the Runge-Kutta methods, receive that in the class of Adams methods there is not method having the property of implicit Runge-Kutta method. Here will show that such methods exist in the class of Multistep methods with constant coefficients. To confirm what has been said, let us consider the following paragraph.

2 On some properties of the forward-jumping methods

For the demonstrated the above noted, let us consider to the following Multistep Method with constant coefficients:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}, \ n = 0, 1, .., N - k. \tag{2.1}$$

From this method one can be receive the Adams and many others similar methods. For example if $\beta_k = 0$, then from the one can be receive Adams-Moulton methods. However, in the case $\beta_k \neq 0$ from the method (2.1) follows Adams-Bashforth methods. Method (2.1) has investigated by many authors (see for example [15],[18, 30, 43], [46]-[61]).

Dahlquist fully investigated method (2.1) and define the conditions imposed on the coefficients, which can be presented as follows:

A. The coefficients α_i, β_i $(i = 0, 1, .., k)$ are the real numbers and $\alpha_k \neq 0$.

B. The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i; \ \delta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i$$

have not common factor different from constant.

C. The following take place: $\rho'(1) \neq 0$, $p > 0$.

Here, by the p-has denoted the order of accuracy for the method (2.1).

Here suppose that he condition A-C are holds.

The multistep methods are compare by using the conception stability and degree, which can been define as follows.

Definition 2.1. Method (2.1) is called as the stable if the roots of the polynomial located in the unit circle, on the boundary of which there is not multiple roots.

Definition 2.2. Integer value p is called as the degree for the method (2.1), if the following asymptotic equality takes place:

$$\sum_{i=0}^{k} (\alpha_i y(x + ih) - h\beta_i y'(x + ih)) = O(h)^{p+1}, \quad h \to 0. \tag{2.2}$$

It is obvious that to determine the accuracy of the method (2.1), it is sufficient to determine the value of the p in the asymptotic equality (2.2).

For the estimation of the degree p for the method (2.1), Dahlquist prove the following theorem.

Theorem 2.3. *(Dahlquist). Suppous that method* (2.1) *has the degree of* p *and stable. Then it does:*

$$p \leq 2[k/2] + 2.$$

And for the each k*, there exist stable method with the degree* $p_{\max} = 2[k/2] + 2$.

For the construction more exact stable methods, Ibrahimov suggested to use forward-jumping (advanced) method, which can be presented as follows:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}; \quad n = 0, 1, 2, ..., k; \quad m > 0. \tag{2.3}$$

It is obvious that in the case $\alpha_{k-m} \neq 0$ and $\sum_{j=k-m+1}^{k} |\beta| \neq 0$ the classes of methods (2.1) and (2.3) do not intersect.

Therefore, each of them is an independent object of research. Ibrahimov has investigated method (2.3) and prove the following theorem:

Theorem 2.4. *(Ibrahimov). If method (2.3) is stable and has the degree of p, then in the class of (2.3), there are methods with the degree*

$$p \leq k+m+1, \quad (k \geq 3).$$

He, constructed stable method of type (2.3) with the degree $p=5$ for the case $k=3$. By the results of Dahlquist, receive that in the class of methods (2.1) there is not stable method with the degree $p>k+1$.
Thus, receive that, these methods are perspective. Now let us to consider comparison method (1.4) with the method of (2.3). For this aim let us to consider case $m=1$ and $k=3$. In this case, from the method (2.3) one can be receive the following method of advanced type:

$$y_{n+2} = (11y_n + 8y_{n+1})/19 + h(10f_n + 57f_{n+1} + 24f_{n+2} - f_{n+3})/57, \tag{2.4}$$

local truncation error for which can be presented as:

$$R_n = -11h^6 y_n^{(6)}/3420 + O(h)^7.$$

Let us to construct method of type (2.1) for the case $k=3$ with the maximum degree, then one can receive:

$$y_{n+3} = y_{n+2} + h(9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n)/24. \tag{2.5}$$

By simple comparison methods (2.4) and (2.5), receive that advanced (forward-jumping) methods are better
If in the method, (1.4) take $m=2$, then receive implicit Runge-Kutta method, which resembles the method (2.4). Note that the Runge-Kutta method can applied to solve some problems in the segment $[x_n, x_{n+1}]$ (see foe example [62]-[76]). However, multistep methods of type (2.1) can be applied to solve any problems in the segment $[x_n, x_{n+1}]$. Methods receive from the method (2.1) for the $k=1$ can compares with Runge-Kutta method and Multistep methods are not the same. However, there are some similarities between them. For the illustration of this, let us to consider the Runge-Kutta method of forth order, which can presented as follows:

$$y_{n+1} = y_n + h(K_1 + 2K_2 + 2K_3 + K_4)/6, \tag{2.6}$$

here, the quantities $k_j (j = 1,2,3,4)$ are determined by the following form:

$$K_1 = f(x_n, y_n); \;\; K_2 = f(x_n + h/2, y_n + hK_1/2);$$
$$K_3 = f(x_n + h/2,\, y_n + hK_2/2); \;\; K_4 = f(x_n + h,\, y_n K_3).$$

If the function of $f(x,y)$ independent from the arguments $y(f(x,y) = \varphi(x)$is holds), then method (2.6) can be written as the following:

$$y_{n+1} = y_n + h(\varphi_n + 4\varphi_{n+1/2} + \varphi_{n+1})/6. \tag{2.7}$$

Let us replace h by the $2h$, then from method (2.7) receive:

$$y_{n+2} = y_n + h(\varphi_n + 4\varphi_{n+1} + \varphi_{n+2})/3. \tag{2.8}$$

This method is the known Simpson method.
And now let us to consider application of Adams methods to calculation following function:

$$y(x) = y(x_0) + \int_{x_0}^{x} \varphi(s)ds, \quad x_0 \leq x \leq X. \tag{2.9}$$

Let us in the (2.9) to put $x = x_k$. Then receive

$$y(x_k) = y(x_0) + h\sum_{i=0}^{k} \beta_i \varphi_{n+i} + R_n. \tag{2.10}$$

Here R_n-is the reminder term. Discarding the remainder term, receive the following:

$$y_{n+k} = y_n + h\sum_{i=0}^{k}\beta_i\varphi_{n+i}. \tag{2.11}$$

This method is also called as the Adams method. It is obvious that method (2.11) more general, than the method (1.5). In the class of methods (2.11), there are stable method with degree $p = k + 2$. For the fairness of this it is enough to recall the Simpsons's method which written as method (2.7).
Let us method (2.11) to write in the following form:

$$y_{n+k} = y_n + h\sum_{i=0}^{k}\beta_i f_{n+i},\ n = 0, 1, .., N - k. \tag{2.12}$$

Considering that the roots of the polynomial $\rho(\lambda) = \lambda^k - 1$ located on the boundary of the unite circle and among of them there are no multiple roots. There for method of (2.12) is stable. It is known that, the roots of the polynomial

$$\lambda^k - (\lambda^{k-1} + \lambda^{k-2} + ... + \lambda + 1)/k,$$

satisfies the conditions of stability, therefore some authors the linear part of Multistep methods selected in the form:

$$y_{n+k} - (y_{n+k-1} + ... + y_{n+1} + y_n)/k.$$

3 Numerical results

For the illustration results received here, let us consider the following example:

$$y' = \lambda y, \quad y(0) = 1, \quad 0 \le x \le 1, \tag{3.1}$$

the exact solution for which can be presented as: $y(x) = \exp(x)$.

As was noted above for the application of advanced methods are arises some difficult for the calculation of the value $y_{n+k-m+1}$, $y_{n+k-m+2}$ etc. It is obvious that in application of the method (2.4) to solve example (3.1) arise necessity calculation the value $y_{n+k-m+1}$. For this one can be used the following method:

$$\hat{y}_{n+3} = y_{n+2} + h(23f_{n+2} - 16f_{n+1} + 5f_n)/24,$$

$$\bar{y}_{n+3} = y_{n+2} + h(9\hat{f}_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n)/24,$$

$$y_{n+2} = (8y_{n+1} + 11y_n)/19 + h(10f_n + 57f_{n+1} + 24f_{n+2} - \bar{f}_{n+3})/57,$$

here $\hat{f}_m = f(x_m, \hat{y}_m)$ and $\bar{f}_m = f(x_m, \bar{y}_m)$.

This algorithm has applied to solve problem (1.1) for the values $\lambda = \pm 1$; $\lambda = \pm 5$. The receiving results are tabulated in the table 1.

Table 1. Results for the step-size $h = 0.1$:

λ	x_n	Error for method (2.3)
$h = 0.1$		
$\lambda = 1$	0.1 0.4 0.7 1.0	1.28E-10 7.27E-10 1.72E-9 3.4E-9
$\lambda = -1$	0.1 0.4 0.7 1.0	1.04E-10 3.24E-10 4.23E-10 4.49E-10
$\lambda = 5$	0.1 0.4 0.7 1.0	4.23E-8 3.96E-8 1.55E-8 4.98E-9
$\lambda = -5$	0.1 0.4 0.7 1.0	1.19E-7 2.24E-6 1.77E-5 1.13E-4

The obtained result corresponds to the theoretical.

4 Conclusion

Let us note that the purpose of the research numerical solution of the initial-value problem for Ordinary Differential Equation by the compares the known numerical methods that were applied to solve above named problem for ODEs started from the end of XVIII century. Methods of type Runge-Kutta development in the XIX century, however, research of Multistep methods also began from XIX century, but with a large map of 40 years. Here also is shown the development of the Numerical methods that were used in solving initial-value problem for ODEs. Also here has defined the relation between the Runge-Kutta methods with the class Multistep Methods. Find some points, where these methods are intersecting. One of them is the Simpson's method. Have given some recommendation for the construction with the best properties. Have defined some connection between of the Runge-Kutta and Multistep Methods. For the comparison of the Runge-Kutta methods with the Multistep Methods in fully from here have used advanced (forward-jumping) methods, which is fully investigated by Ibrahimov. We hope that the results obtained, here will help many specialists who are engaged in the field of numerical methods. The evolution towards advanced methods, such as those outlined by Ibrahimov, unveils new avenues for exploration, revealing deeper connections between the classes of techniques. Their performance in specific scenarios, particularly regarding stability and convergence, provides crucial insights for practitioners. As the field progresses, it is critical to embrace these interrelationships, optimizing existing methods, and seeking innovations that continue to advance the numerical resolution of ODEs, ultimately benefiting diverse applications across scientific and engineering disciplines. Through comparative examples, one can observe that specific initial conditions yield congruent results from both the Adams and Runge-Kutta approaches, affirming their structural and functional relationship in numerical computations. The versatility of these methods underscores their significance in scientific and engineering fields, where the demand for accurate and efficient solutions to ODEs remains paramount.

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