Karshi Multidisciplinary International Scientific Journal

Vol. 1(2) (2024), 155–164 © Research Expansion Alliance (REA) 2024

THE ANALYTICAL FUNCTION DEFINED IN THE HAMILTON-JACOBI & SCHRÖDINGER APPROACH AND THE CLASSICAL SCHRÖDINGER EQUATION

J.D. Bulnes, M.A.I. Travassos, D.A. Juraev and J.L. López-Bonilla

Communicated by Nazira Mohubbat Mammadzada

Abstract: In this paper, we show that a specific extension of the Schrödinger ansatz for two free particles, under the requirement that this function (ansatz) be analytical, is compatible with the expected physical result for a quantum description at the quantum-classical boundary; that is, its total erasure in the transition to a description compatible with classical physics. Using the Cauchy-Riemann relation, the Laplace equation, and the Hamilton-Jacobi equation, we have shown that this function verifies a classical equation arising from the time-dependent Schrödinger equation, since in the assumed context the time variable can be taken as a parameter since it is irrelevant in the process of approximation to the quantum-classical boundary.

Keywords and phrases: Schrödinger Ansatz, Hamilton-Jacobi Equation, Analytic Function, Laplace Equation, Classical Schrödinger Equation.

MSC 2010 Classifications: Primary 70H20; Secondary 70F17.

1 Introduction

The problem Newton solved by discovering the force of gravitational attraction, which explains the kinematic laws of planetary motion, discovered by Kepler, is called the *inverse problem* [1]-[2]. On the other hand, the problem solved (or rather, the approach he followed) by Schrödinger, starting from an ansatz to deduce a differential equation for it, also corresponds to an inverse problem (approach) [3]-[4]. Inverse problems, which are very relevant in various fields [5]-[12], have the general characteristic of not having a unique solution. In Schrödinger's problem, it is clear that arising from this general approach, one can, in principle, find mathematical solutions distinct from Schrödinger's equation. Therefore, it is interesting to ask whether any new equation arising from this approach also carries physical content? Would these equations be related to Schrödinger's? This article develops an example of this situation.

1.1 Brief review

We know that the approach that uses: (i) the so-called Schrödinger ansatz, which, in the case with a spatial variable, is written as:

$$\psi(x,t) = exp\Big\{(i/\hbar)S(x,t)\Big\}, \tag{1.1}$$

where $S(x,t)$ is the action function for a physical system, (ii) the Hamilton-Jacobi equation, (iii) as well as a certain conceptual context, enabled Prof. Schrödinger to obtain his famous equation. In several articles on the same problem, using the classical Hamilton-Jacobi equation, the approximation that consists of taking the limit $\hbar \to 0$ to eliminate an excess term [13]-[17] is used in the approximate passage to the Schrödinger equation. Only recently, it was shown, for the case of quantum systems in one spatial dimension and when the acting potential does not explicitly depend on the time variable, that one can exactly deduce both the complete Schrödinger equation from the classical Hamilton-Jacobi equation, as well as it's inverse [18]. In this article, considering the action $S(x_1, x_2, t)$ for a physical system formed by two free particles of equal mass, with coordinate variables x_1 and x_2, separately, which are moving in opposite directions, we will determine under which condition, in the form of a differential equation for $C(x_1 - x_2)$, the following function:

$$\Psi(x_1, x_2, t) = C(x_1 - x_2)\ exp\Big\{(i/\hbar)S(x_1, x_2, t)\Big\}, \tag{1.2}$$

will correspond to an analytical function. The expression in (1.2), which besides being an obvious extension of (1.1), is only formal since $C(x_1 - x_2)$ is not defined. By finding $C(x_1 - x_2)$, we will determine the corresponding mathematical consequences for $\Psi(x_1, x_2, t)$; in particular, for "infinite" separation between the particles: $(x_1 - x_2 \to \infty)$. In this context, since we will not use the usual ansatz (1.1), it is obvious that there is no guarantee that function (1.2) verifies the Schrödinger equation.

Finally, in recent literature, we find several tools, approaches, and formal problems in mathematical physics [19]-[26] and the identification and resolution of new quantum problems [27]-[40] that can be used in new extensions and new contexts for quantum mechanics.

2 Mathematical developments

Just for simplicity of notation, we will use here $\Phi(x_1, x_2, t)$ instead of writing $(1/\hbar)\ S(x_1, x_2, t)$. Since the sought function $C(x_1 - x_2)$ must lead to an analytic function $\Psi(x_1, x_2, t)$, the real and imaginary parts of the latter must be required to satisfy the Cauchy-Riemann conditions[1] [42]; from which certain relations between $C(x_1 - x_2)$ and $\Phi(x_1, x_2, t)$ will follow.

On the other hand, as is well known, the real and imaginary parts of an analytic function are harmonic functions, that is, they verify Laplace's equation [43], as long as these parts have continuous derivatives up to and including second-order. The above, together with the Hamilton-Jacobi equation [17], will produce a complementary equation, which together with the first relations found via Cauchy-Riemann's conditions, will lead to the equation for $C(x - y)$. Let us see this.

Using simplified notation, function (1.2) is written as:

$$\Psi(x_1, x_2, t) = C(x_1 - x_2)e^{i\Phi(x_1,x_2,t)}\ , \tag{2.1}$$

2.1 Relations arising from the Cauchy-Riemann's condition

The real and imaginary parts of (2.1), which are directly identifiable:

$$\Psi(x_1, x_2, t) = C(x_1 - x_2)\ cos\Phi(x_1, x_2, t)\ +\ i\ C(x_1 - x_2)\ sin\Phi(x_1, x_2, t),$$

$$\Rightarrow\quad u(x_1, x_2, t) = C(x_1 - x_2)\ cos\Phi(x_1, x_2, t),\quad v(x_1, x_2, t) = C(x_1 - x_2)\ sin\Phi(x_1, x_2, t) \tag{2.2}$$

will not be independent, but will be connected by the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2} \quad \& \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}. \tag{2.3}$$

Here it is sufficient to consider only the first of them. Substituting (2.2) in (2.3) we have:

$$\frac{\partial}{\partial x_1}\Big\{C(x_1 - x_2)cos\Phi\Big\} = \frac{\partial}{\partial x_2}\Big\{C(x_1 - x_2)sin\Phi\Big\}. \tag{2.4}$$

We change the variable $\eta \equiv x_1 - x_2$, and then we derive the expressions in (2.4), obtaining:

$$C(\eta)\Big(-sin\Phi\Big)\frac{\partial\Phi}{\partial x_1} + \frac{dC}{d\eta}\frac{d\eta}{dx_1}\ cos\Phi\ =\ C(\eta)\Big(cos\Phi\Big)\frac{\partial\Phi}{\partial x_2} + \frac{dC}{d\eta}\frac{d\eta}{dx_2}\ sin\Phi. \tag{2.5}$$

Or also,

$$-C(\eta)\ \frac{\partial\Phi}{\partial x_1}\ sin\Phi + \frac{dC}{d\eta}(\eta)\ cos\Phi\ =\ C(\eta)\ \frac{\partial\Phi}{\partial x_2}\ cos\Phi - \frac{dC}{d\eta}(\eta)\ sin\Phi. \tag{2.6}$$

---

[1] Also known as Euler-d'Alembert relations [41].

Grouping terms appropriately we have:

$$\Big(- C(\eta)\,\frac{\partial \Phi}{\partial x_1}(x_1,x_2) + \frac{dC}{d\eta}(\eta)\Big) sin\Phi + \Big(\frac{dC}{d\eta}(\eta) - C(\eta)\,\frac{\partial \Phi}{\partial x_2}(x_1,x_2)\Big) cos\Phi = 0. \tag{2.7}$$

And being the functions $sin\Phi$ and $cos\Phi$ linearly independent for all $x_1, x_2 \in \Re$, we obtain, from (2.7), the independent equations:

$$\frac{dC}{d\eta} = C(\eta)\,\frac{\partial \Phi}{\partial x_1} \quad \& \quad \frac{dC}{d\eta} = C(\eta)\,\frac{\partial \Phi}{\partial x_2}. \tag{2.8}$$

After equating the equations in (2.8), we obtain:

$$\frac{\partial \Phi}{\partial x_1} - \frac{\partial \Phi}{\partial x_2} = 0, \tag{2.9}$$

Where it was previously considered that $C(x_1 - x_2)$ is non-zero, as it should be. Then, from the equation on the left-hand side in (2.8) it follows, by differentiation with respect to the variable x_1 (assigned to the first particle), that:

$$\frac{d^2C}{d\eta^2}\,\frac{d\eta}{dx_1} = \Big(\frac{dC}{d\eta}\,\frac{d\eta}{dx_1}\Big)\frac{\partial \Phi}{\partial x_1} + C(\eta)\,\frac{\partial^2 \Phi}{\partial x_1^2}. \tag{2.10}$$

Or also,

$$\frac{d^2C}{d\eta^2} = \frac{dC}{d\eta}\,\frac{\partial \Phi}{\partial x_1} + C(\eta)\,\frac{\partial^2 \Phi}{\partial x_1^2}. \tag{2.11}$$

And from the equation on the right-hand side in (2.8) it follows, by differentiation with respect to the variable x_2:

$$\frac{d^2C}{d\eta^2}\,\frac{d\eta}{dx_2} = \Big(\frac{dC}{d\eta}\,\frac{d\eta}{dx_2}\Big)\frac{\partial \Phi}{\partial x_2} + C(\eta)\,\frac{\partial^2 \Phi}{\partial x_2^2}. \tag{2.12}$$

Or also,

$$-\frac{d^2C}{d\eta^2} = -\frac{dC}{d\eta}\,\frac{\partial \Phi}{\partial x_2} + C(\eta)\,\frac{\partial^2 \Phi}{\partial x_2^2}. \tag{2.13}$$

Adding equations (2.11) and (2.13) we obtain:

$$\frac{dC}{d\eta}\Big\{\frac{\partial \Phi}{\partial x_1} - \frac{\partial \Phi}{\partial x_2}\Big\} + C(\eta)\Big\{\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2}\Big\} = 0. \tag{2.14}$$

Considering the result in (2.9) we have that (2.14) simplifies to:

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} = 0, \tag{2.15}$$

Where it has been previously considered that $C(x - y)$ is non-zero. Thus, $\Phi(x, y)$ satisfies Laplace's equation and the equation in (2.9). It can also be said, in the context being considered and according to the notation used, that the action S satisfies Laplace's equation.

2.2 Relations arising from the Laplace and Hamilton-Jacobi's equations

Consider Laplace's equation for the imaginary part of $\Psi(x_1, x_2, t)$, that is:

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = 0. \tag{2.16}$$

Then, substituting the expression for $v(x_1, x_2, t)$, which is deduced from (2.2), in (2.16), we have:

$$\frac{\partial^2}{\partial x_1^2}\Big(C(\eta)\ sin\Phi\Big) + \frac{\partial^2}{\partial x_2^2}\Big(C(\eta)\ sin\Phi\Big) = 0 \tag{2.17}$$

Performing the first derivative on each term in (2.17) we have expressly:

$$\frac{\partial}{\partial x_1}\left\{C(\eta)\ cos\ \Phi\ \frac{\partial\Phi}{\partial x_1}+\frac{dC}{d\eta}\ sin\ \Phi\right\}+\ \frac{\partial}{\partial x_2}\left\{C(\eta)\ cos\ \Phi\ \frac{\partial\Phi}{\partial x_2}-\frac{dC}{d\eta}\ sin\ \Phi\right\}\ =\ 0 \quad (2.18)$$

In (2.18), after performing the second derivative on each term, we get:

$$C(\eta)\left\{-\ sin\ \Phi\ \left(\frac{\partial\Phi}{\partial x_1}\right)^2+cos\ \Phi\ \left(\frac{\partial^2\Phi}{\partial x_1^2}\right)\right\}+2\frac{dC}{d\eta}\ \frac{\partial\Phi}{\partial x_1}\ cos\ \Phi+2\frac{d^2C}{d\eta^2}\ sin\ \Phi+$$

$$+\ C(\eta)\left\{cos\ \Phi\ \left(\frac{\partial^2\Phi}{\partial x_2^2}\right)-sin\ \Phi\ \left(\frac{\partial\Phi}{\partial x_2}\right)^2\right\}-2\frac{dC}{d\eta}\ \frac{\partial\Phi}{\partial x_2}\ cos\ \Phi=0. \quad (2.19)$$

Arranging terms in (2.19) we have:

$$-\ C(\eta)\ sin\ \Phi\left[\left(\frac{\partial\Phi}{\partial x_1}\right)^2+\left(\frac{\partial\Phi}{\partial x_2}\right)^2\right]+C(\eta)\ cos\ \Phi\left[\left(\frac{\partial^2\Phi}{\partial x_1^2}\right)+\left(\frac{\partial^2\Phi}{\partial x_2^2}\right)\right]+$$

$$+\ 2\frac{d^2C}{d\eta^2}\ sin\ \Phi+2\frac{dC}{d\eta}\ cos\ \Phi\left(\frac{\partial\Phi}{\partial x_1}-\frac{\partial\Phi}{\partial x_2}\right)=0. \quad (2.20)$$

Using the results (2.15) and (2.9), equation (2.20) reduces to:

$$\left\{\ \frac{d^2C}{d\eta^2}-\ \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x_1}\right)^2+\left(\frac{\partial\Phi}{\partial x_2}\right)^2\right]C(\eta)\ \right\}\ sin\ \Phi=0. \quad (2.21)$$

That is, we have:

$$\frac{d^2C}{d\eta^2}-\ \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x_1}\right)^2+\left(\frac{\partial\Phi}{\partial x_2}\right)^2\right]C(\eta)=0. \quad (2.22)$$

Now it is convenient to return to the original symbology, $\Phi(x_1,x_2,t)\equiv(i/\hbar)S(x_1,x_2,t)$, to place the action in (2.22); then, we write:

$$\frac{d^2C}{d\eta^2}-\ \frac{1}{2\hbar^2}\left[\left(\frac{\partial S}{\partial x_1}\right)^2+\left(\frac{\partial S}{\partial x_2}\right)^2\right]C(\eta)=0. \quad (2.23)$$

On the other hand, from the Hamilton-Jacobi equation [14], assuming that the two particles are free and have (the same parameter of) mass m, one can write:

$$\frac{1}{2}\left[\left(\frac{\partial S}{\partial x_1}\right)^2+\left(\frac{\partial S}{\partial x_2}\right)^2\right]=-\ m\ \frac{\partial S}{\partial t}(x_1,x_2,t). \quad (2.24)$$

When the physical system does not have a potential that explicitly depends on time, as in the case with two free particles, one can write the (non-standard) action [17] in an additive and separable manner in its variables:

$$S(x_1,x_2,t)=-at+g(x_1,x_2),\quad a\neq 0. \quad (2.25)$$

Where the corresponding equation for the unknown function $g(x_1,x_2)$ is obtained directly from the Hamilton-Jacobi equation, see (2.31). Thus, we have that (2.24) can be rewritten as:

$$\frac{1}{2}\left[\left(\frac{\partial S}{\partial x_1}\right)^2+\left(\frac{\partial S}{\partial x_2}\right)^2\right]=ma. \quad (2.26)$$

2.3 (Quantum) Equation for $C(x_1-x_2)$

Substituting (2.26) into (2.23) we finally find the equation for $C(\eta)$:

$$\frac{d^2C}{d\eta^2}-\ \frac{ma}{\hbar^2}\ C(\eta)=0. \quad (2.27)$$

Note that the above equation, which is quantum (due to the presence of $\hbar$), linear and dimensionally correct, has as a general solution:

$$C(x_1 - x_2) = A\ exp\Big\{\frac{\sqrt{ma}}{\hbar}(x_1 - x_2)\Big\} + B\ exp\Big\{-\frac{\sqrt{ma}}{\hbar}(x_1 - x_2)\Big\}, \tag{2.28}$$

but, the physical solution corresponds to taking $A = 0$, since an infinite growth in the value of a quantum function (in this case, when $x_1 - x_2 \to \infty$) is not physically acceptable [44]; thus, we have that (2.28) can be reduced to:

$$C(x_1 - x_2) = B\ exp\Bigg\{-\frac{\sqrt{ma}}{\hbar}(x_1 - x_2)\Bigg\}. \tag{2.29}$$

Consequently, the function $\Psi(x_1, x_2, t)$, in (1.2), is formally written as:

$$\Psi(x_1, x_2, t) = B\ exp\Bigg\{-\frac{\sqrt{ma}}{\hbar}(x_1 - x_2)\Bigg\}\ exp\Bigg\{(i/\hbar)\Big(-at + g(x_1, x_2)\Big)\Bigg\}. \tag{2.30}$$

But, in (2.30), the function $g(x_1, x_2)$ remains undefined. This function must verify the following equation in partial derivatives, non-linear and non-homogeneous:

$$\left(\frac{\partial g}{\partial x_1}\right)^2 + \left(\frac{\partial g}{\partial x_2}\right)^2 = 2ma\ , \tag{2.31}$$

Whose solution is found directly:

$$g(x_1, x_2) = \epsilon\sqrt{ma}\ (x_1 + x_2)\ ,\ \ \epsilon = \pm 1. \tag{2.32}$$

Then, the total function, describing the state for two particles, is written as:

$$\Psi(x_1, x_2, t) = B\ exp\Big\{-\frac{\sqrt{ma}}{\hbar}(x_1 - x_2)\Big\}\ exp\Big\{(i/\hbar)\Big(\epsilon\sqrt{ma}\,(x_1 + x_2) - at\Big)\Big\}. \tag{2.33}$$

The amplitude of $\Psi(x_1, x_2, t)$, that is, the non-oscillating factor, progressively attenuates as the physical system moves. As a result, function (2.33) vanishes for infinite separation between the particles:

$$\text{When: } (x_1 - x_2) \to \infty \ \text{ then: } \Psi(x_1, x_2, t) \to 0. \tag{2.34}$$

That is, a manifestation of the quantum description, the constant $\hbar$, previously present in this function, disappears. In this way, the analytical function in (2.33), under the mathematical condition expressed by the limit $(x_1 - x_2) \to \infty$, which should correspond to the classical limit of quantum mechanics, vanishes at the quantum-classical boundary[2]. Which equation is satisfied by function (2.33)? This will be considered in section 2.5. Let us first look at an aspect of the mathematical compatibility of function (2.33), which should correspond, by hypothesis, to the total state of two free particles.

2.4 More on the partial plane waves for each particle

For free particles that move independently of each other, their total function should correspond to the product of the functions of each particle. We will see below that function (2.33) can be decomposed as a product of functions:

$$\Psi(x_1, x_2, t) = \psi_1(x_1, t)\ \psi_2(x_2, t). \tag{2.35}$$

(Here we are mainly interested in compatibility and not in univocal[3] identification) being the expressions of the partial functions:

$$\psi_1(x_1, t) = B^{1/2}\ exp\Big\{-\frac{\sqrt{ma}}{\hbar}\ x_1\Big\}\ exp\Big\{(i/\hbar)\Big(\epsilon\sqrt{ma}\ x_1 - \frac{a}{2}t\Big)\Big\}, \tag{2.36}$$

---

[2] Of which, according to the formalism of quantum mechanics, would not be a collapse of the function.

[3] Assuming that $\Psi(x_1, x_2, t)$ in (2.33) verifies a linear equation, we also have the possibility of writing: $\Psi(x_1, x_2, t) = \psi_1(x_1, t)\ \psi_2(x_2, t)|_{\epsilon=+1} + \psi_1(x_1, t)\ \psi_2(x_2, t)|_{\epsilon=-1} = 2\psi_1(x_1, t)\ \psi_2(x_2, t)$.

$$\psi_2(x_2,t) = B^{1/2}\, exp\Big\{\, \frac{\sqrt{ma}}{\hbar}\, x_2\Big\}\, exp\Big\{(i/\hbar)\Big(\epsilon\sqrt{ma}\, x_2 - \frac{a}{2}t\Big)\Big\}, \tag{2.37}$$

Where we correctly note, in the phase angle in (2.36) and (2.37), the coefficient $(a/2)$ in the time-dependent term, which corresponds to half the energy of the complete system, a. Assuming that the particles move away from each other from the common origin ($x_1 = x_2 = 0$) of the reference direction, we would have to take as a consequence[4] $x_2 = -x_1$. Therefore, considering $x_1 > 0$ and choosing $\epsilon = 1$, we have:

$$\psi_1(x_1,t) = B^{1/2}\, exp\Big\{ -\frac{\sqrt{ma}}{\hbar}\, x_1\Big\}\, exp\Big\{i\Big(\frac{\sqrt{ma}}{\hbar}\, x_1 - \frac{a}{2\hbar}t\Big)\Big\}\,, \tag{2.38}$$

$$\psi_2(-x_1,t) = B^{1/2}\, exp\Big\{ -\frac{\sqrt{ma}}{\hbar}\, x_1\Big\}\, exp\Big\{ -i\Big(\frac{\sqrt{ma}}{\hbar}\, x_1 + \frac{a}{2\hbar}t\Big)\Big\}\,. \tag{2.39}$$

Note that the two functions have the same amplitude and that this attenuates with the increase of x_1. For the functions described above, we have that the following definitions:

$$k_1 = k_2 \equiv \sqrt{ma}/\hbar \qquad \& \qquad \omega_1 = \omega_2 \equiv a/2\hbar, \tag{2.40}$$

are dimensionally compatible with the wavenumber (k) and the frequency (ω). Thus, the expressions (2.38) and (2.39) represent, separately, two plane waves that, moving away from each other in opposite directions, attenuate as they propagate and whose product coincides with the function (2.33). We thus verify that the function $\Psi(x_1, x_2, t)$ in (2.33) is compatible with the initial assumption that this function corresponds to two free particles.

2.5 A classical equation for $\Psi(x_1, x_2, t)$

We can directly obtain the equation for the function Ψ by taking advantage of the fact that its real and imaginary parts, in (2.2), must verify Laplace's equation, separately. However, to highlight a specific relationship between the equation for $\Psi(x_1, x_2, t)$ and the equation for $C(x_1 - x_2)$, we will go another way. In particular, it is interesting to verify whether the equation for Ψ is related to the Schrödinger equation.

From the formal expression in (1.2) we have, by derivation in relation to the variable x_1 that:

$$\frac{\partial\Psi}{\partial x_1} = C(\eta)\Big\{\frac{i}{\hbar}\,\frac{\partial S}{\partial x_1} exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}\Big\} + \frac{dC}{d\eta}\,\frac{d\eta}{dx_1}\, exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}, \tag{2.41}$$

A second derivation, in relation to the same variable, generates the expression:

$$\begin{aligned}\frac{\partial^2\Psi}{\partial x_1^2} &= C(\eta)\,\Big(-\frac{1}{\hbar^2}\Big)\Big(\frac{\partial S}{\partial x_1}\Big)^2 exp\Big\{iS(x_1,x_2,t)/\hbar\Big\} + \\ &+ C(\eta)\,\frac{i}{\hbar}\Big(\frac{\partial^2 S}{\partial x_1^2}\Big) exp\Big\{iS(x_1,x_2,t)/\hbar\Big\} + \frac{dC}{d\eta}\,\frac{d\eta}{dx_1}\,\frac{i}{\hbar}\Big(\frac{\partial S}{\partial x_1}\Big) exp\Big\{iS(x_1,x_2,t)/h\Big\} + \\ &+ \frac{dC}{d\eta}\,\frac{i}{\hbar}\Big(\frac{\partial S}{\partial x_1}\Big) exp\Big\{iS(x_1,x_2,t)/\hbar\Big\} + \frac{d^2C}{d\eta^2}\, exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}.\end{aligned} \tag{2.42}$$

And, after two derivations of (1.2) in relation to the variable x_2, we have:

$$\begin{aligned}\frac{\partial^2\Psi}{\partial x_2^2} &= C(\eta)\,\Big(-\frac{1}{\hbar^2}\Big)\Big(\frac{\partial S}{\partial x_2}\Big)^2 exp\Big\{iS(x_1,x_2,t)/h\Big\} + \\ &+ C(\eta)\,\frac{i}{\hbar}\Big(\frac{\partial^2 S}{\partial x_2^2}\Big) exp\Big\{iS(x_1,x_2,t)/\hbar\Big\} - \frac{dC}{d\eta}\,\frac{i}{\hbar}\Big(\frac{\partial S}{\partial x_2}\Big) exp\Big\{iS(x_1,x_2,t)/\hbar\Big\} + \\ &- \frac{dC}{d\eta}\,\frac{i}{\hbar}\Big(\frac{\partial S}{\partial x_2}\Big) exp\Big\{iS(x_1,x_2,t)/\hbar\Big\} + \frac{d^2C}{d\eta^2}\, exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}.\end{aligned} \tag{2.43}$$

---

[4] We must not forget that the variables x_1 and x_2 are independent for $\Psi(x_1, x_2, t)$, for $C(x_1 - x_2)$, as well as for $g(x_1, x_2)$.

Adding the expressions (2.42) and (2.43) we have:

$$\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}=$$

$$=-\frac{1}{\hbar^2}\left\{\left(\frac{\partial S}{\partial x_1}\right)^2+\left(\frac{\partial S}{\partial x_2}\right)^2\right\}\Psi+\frac{i}{\hbar}\left\{\frac{\partial S}{\partial x_1}-\frac{\partial S}{\partial x_2}\right\}\frac{dC}{d\eta}\,exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}+$$

$$+\frac{i}{\hbar}\left\{\frac{\partial^2 S}{\partial x_1^2}+\frac{\partial^2 S}{\partial x_2^2}\right\}\Psi+\,2\frac{d^2C}{d\eta^2}\,exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}. \tag{2.44}$$

Using the results (2.9) and (2.15), the expression (2.44) is simplified, arriving at:

$$\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}=-\frac{1}{\hbar^2}\left\{\left(\frac{\partial S}{\partial x_1}\right)^2+\left(\frac{\partial S}{\partial x_2}\right)^2\right\}\Psi+2\frac{d^2C}{d\eta}\,exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}. \tag{2.45}$$

Substituting (2.26) in (2.45), we have,

$$\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}=-\frac{2ma}{\hbar^2}\,\Psi+2\frac{d^2C}{d\eta}\,exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}. \tag{2.46}$$

Or also, considering (1.2), we have:

$$\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}=2\left\{\frac{d^2C}{d\eta^2}(\eta)-\frac{ma}{\hbar^2}\,C(\eta)\right\}exp\Big\{iS(x_1,x_2,t)/\hbar\Big\}. \tag{2.47}$$

The term that appears in the center in (2.47) is precisely the one that appears on the left side in equation (2.27); thus, in (2.47), it must be zero. Therefore, we arrive at the expected result:

$$\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}=0\,, \tag{2.48}$$

Which is a classical equation (does not contain $\hbar$). It is a surprise that, since Ψ is a quantum function, it verifies a classical equation. From (2.48), no solution that carries time dependence is expected; however, Ψ, em (2.33), depends, in addition to the coordinate variables x_1 and x_2, on the time variable t and fulfills equation (2.48). How can this be explained?

2.6 A plausible interpretation for (2.48)

The fact that function (2.33), verifying equation (2.48), presents time dependence, can be interpreted as corresponding to a situation in which this equation comes from a larger equation, with a time derivative term.

In the problem considered here, a special context or aspect of the physical manifestation can be identified, where the temporal dependence is not physically relevant, even though the corresponding "variable" remains in the wave function. Specifically, it is observed, from a physical point of view, that there is no difference between reaching the quantum-classical boundary quickly or slowly. Reaching this boundary determines the "collapse" of a quantum description, which is the same regardless of how quickly this happened. In this scenario, due to the essence of the process, the time variable loses meaning, has no relevance, and only remains as a parameter, so that:

$$\Psi(x_1,x_2,t)\equiv\Psi(x_1,x_2;t),\quad\Rightarrow\quad\frac{\partial\Psi}{\partial t}=0. \tag{2.49}$$

Thus, the complete Schrödinger equation for two free quantum particles (with the same mass parameter m), in the case of 1-dimensional motion:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}\right)=i\hbar\frac{\partial\Psi}{\partial t}. \tag{2.50}$$

is reduced to the classical equation (2.48) in the situation of proximity to the classical-quantum boundary.

Complementarily, and independently of what is developed here, it can be seen in [29] that it was verified that an entangled function with the form:

$$\Psi(\vec{x}_1, \vec{x}_2) = \sum_n C_n \Big(|\vec{x}_2 - \vec{x}_1| / \lambda_C \Big)\; \psi_n(\vec{x}_1)\; u_n(\vec{x}_2), \tag{2.51}$$

which is a generalization of the Einstein-Podolsky-Rosen state [45], where λ_C is the Compton wavelength of the particles involved, $\psi_n(\vec{x}_1)$ eigenfunctions of an arbitrary observable for particle 1, etc., satisfies the 3-dimensional version of equation (2.48) under the main requirement that the function $C_n\Big(|\vec{x}_2 - \vec{x}_1|/\lambda_C\Big)$, for $|\vec{x}_2 - \vec{x}_1| \to \infty$, verifies: $C_n \to 0$; that is, that in (53) the quantum entanglement that carries this in the context of approximation to the quantum-classical boundary is erased.

3 Conclusion

The main result of this work was to have shown that in the proximity of the quantum-classical boundary, corresponding to the condition $x_1 - x_2 \to \infty$, which effectively proves to be (partially) equivalent to the classical limit $\hbar \to 0$, the description for two free quantum particles disappears spontaneously under the sole requirement that the function representing their joint state be analytic. At no stage of the development presented have we required that any functions, such as the harmonic functions u and v, have zero values at the quantum-classical boundary.

This result makes sense, since starting from the Hamilton-Jacobi & Schrödinger approach, specifically from the classical Hamilton-Jacobi equation, it would be reasonable to expect that one can also obtain results corresponding to the “outermost surface" of the microscopic world; that is, the quantum-classical boundary.

What we are calling the classical Schrödinger equation is nothing more than a reduced version of the full (quantum) Schrödinger equation for a situation of approximation to the classical-quantum boundary, in which, effectively, the time variable can be taken as a parameter.

Finally, it may be interesting to point out that the quantum equation for the quantum function $C(x_1 - x_2)$, in (2.27), is the one that leads to the classical equation for the quantum function Ψ, see (2.47). A simple, operational, and provisional way of seeing this consists in assuming a kind of “transfer" of the quantum character of the general Schrödinger equation to the equation for $C(x_1 - x_2)$; thus, the equation for Ψ could no longer be quantum.

References

[1] A.I. Prilepko, D.G. Orlovsky, I.A. Vasin, *Methods for solving inverse problems in mathematical physics*, Marcel Dekker, Inc., New York, (2000).

[2] R.M. Santilli, *Foundations of Theoretical Mechanics I. The Inverse Problem in Newtonian Mechanics*, Springer Science, (1978).

[3] E. Schrödinger, *Collected Papers on Wave Mechanics*, Chelsea, New York, (1978).

[4] L.D. Landau, E.M. Lifshitz, *Quantum Mechanics*, Addison-Wesley, Reading, (1965).

[5] B. Bhalgamiya, M.A. Novotny, Exact solutions for the inverse problem of the time-independent Schrödinger equation, *Am. J. Phys.*, **92:12**, 975–979 (2024).

[6] D.A. Juraev, E.E. Elsayed, J.D. Bulnes, P. Agarwal, R.K. Saeed, History of ill-posed problems and their application to solve various mathematical problems, *Engineering Applications*, **2:3**, 279–290 (2023).

[7] A. Rangel-Merino, R. Linares y Miranda, J. López-Bonilla, Genetic Algorithms and the Inverse Problem of Electromagnetic Radiation, *Journal of the Institute of Engineering*, **8:1**, 231236 (2011).

[8] J.D. Bulnes, M. Cavalcante, M.A.I. Travassos, J.L. López-Bonilla, Una simulación matemática del efecto Zeeman de un campo magnético, *Journal de Ciencia e Ingeniería*, **15:2**, 24–32 (2023).

[9] Y. Fayziev, Q. Buvaev, D. Juraev, N. Nuralieva, Sh. Sadullaeva, The inverse problem for determining the source function in the equation with the Riemann-Liouville fractional derivative, *Global and Stochastic Analysis*, **9:2**, 4352 (2022).

[10] D.A. Juraev, A.A. Tagiyeva, J.D. Bulnes and B. Drumea, On the regularization of the Cauchy problem for matrix factorizations of the Helmholtz equation in R^m, *Karshi Multidisciplinary International Scientific Journal*, **1:2**, 51–68 (2024).

[11] D.A. Juraev, N.M. Mammadzada and M. Israr, On the Application of Ill-posed problems of equations of mathematical physics, *Karshi Multidisciplinary International Scientific Journal*, **1:2**, 43–50 (2024).

[12] I.E. Niyozov, D.A. Juraev, R.F. Efendiev, M.J. Jalalov, The Cauchy Problem for the System of Elasticity, *Journal of Contemporary Applied Mathematics*, **14:2**, 92–107 (2024).

[13] A. Small and K.S. Lam, Simple derivations of the Hamilton-Jacobi equation and the eikonal equation without the use of canonical transformations, *Am. J. Phys.*, **79:6**, 678–681 (2011).

[14] P.R. Sarma, Direct derivation of Schrödinger equation from Hamilton-Jacobi equation using uncertainty principle, *Rom. Journ. Phys.*, **56:10**, 1053-1056 (2010).

[15] L . Kocis, Ehrenfest Theorem for the Hamilton-Jacobi Equation, *Acta Physica Polonica A*, **102:6**, 709–716 (2002).

[16] D. Derbes, Feynman's derivation of the Schrödinger equation, *Am. J. Phys.*, **64:7**, 881–884 (1996).

[17] H. Goldstein, *Classical Mechanics*, Addison Wesley, (1980).

[18] J.D. Bulnes, M.A.I. Travassos, D.A. Juraev, J. López-Bonilla, From the Hamilton-Jacobi equation to the Schrödinger equation and vice versa, without additional terms and approximations, *Himalayan Physics*, **11**, 21–27 (2024).

[19] D.A. Juraev, N.M. Mammadzada, J.D. Bulnes, S.K. Gupta, G.A. Aghayeva, V.R. Ibrahimov, Regularization of the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain, *Mathematics and Systems Science*, **2:2**, 1–16 (2024).

[20] D.A. Juraev, A.A. Tagiyeva, J.D. Bulnes and G.X.-G. Yue, On the solution of the Ill-posed Cauchy problem for elliptic systems of the first order, *Karshi Multidisciplinary International Scientific Journal*, **1:1**, 17–26 (2024).

[21] D.A. Juraev, P. Agarwal, A. Shokri, E.E. Elsayed, Integral formula for matrix factorizations of Helmholtz equation, *Recent trends in fractional calculus and its applications, advanced studies in complex systems*, Elsevier, pp. 123-146 (2024).

[22] D.A. Juraev, P. Agarwal, A. Shokri, E.E. Elsayed, Cauchy problem for matrix factorizations of Helmholtz equation on a plane, *Recent trends in fractional calculus and its applications, advanced studies in complex systems*, Elsevier, pp. 147-175 (2024).

[23] D.A. Juraev, P. Agarwal, A. Shokri, E.E. Elsayed, The Cauchy problem for matrix factorizations of Helmholtz equation in space, *Recent trends in fractional calculus and its applications, advanced studies in complex systems*, Elsevier, pp. 177-210 (2024).

[24] D.A. Juraev, Y.S. Gasimov, On the Regularization Cauchy Problem for Matrix Factorizations of the Helmholtz Equation in a Multidimensional Bounded Domain, *Azerbaijan Journal of Mathematics*, **2:1**, 142 (2022).

[25] D.A. Juraev, Solution of the Ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation on the plane, *Global and Stochastic Analysis*, **8:3**, 1–17 (2021).

[26] D.A. Juraev, P. Agarwal, A. Shokri, E.E. Elsayed, J.D. Bulnes, On the solution of the Ill-posed Cauchy problem for elliptic systems of the first order, *Stochastic Modelling & Computational Sciences*, **3**, 1–21, (2023).

[27] J.D. Bulnes, M.A.I. Travassos, D.A. Juraev, G. Dias and J. López-Bonilla. Mathematical simulation of à la Einstein loss of the non-separable character of a function for two particles at the quantum-classical boundary. *Open Journal of Mathematical Sciences*, **8**, 113–127 (2024).

[28] J.D. Bulnes, M.A.I. Travassos, D.A. Juraev, J.E. Ottoni, J. López-Bonilla, Stern-Gerlach Centennial: Parity, gradient effect, and an analogy with the Higgs field, *Himalayan Physics*, **11**, 68–77 (2024).

[29] J.D. Bulnes, An unusual quantum entanglement consistent with Schrödinger's equation, *Global and Stochastic Analysis*, **9**, 79–87 (2022).

[30] J.D. Bulnes, M.A.I. Travassos, H.E. Caicedo-Ortiz, J. López-Bonilla, Más de un tipo de entrelazamiento en el mundo cuántico y ninguno en el mundo clásico, *Journal de Ciencia e Ingeniería*, **16:1**, 5–12 (2024).

[31] J.D. Bulnes, D.A. Juraev, J.L. López-Bonilla, M.A.I. Travassos, Exact de coupling of a coupled system of two stationary Schrodinger equation, *Stochastic Modelling Computational Sciences*, **3**, 23–28 (2023).

[32] J.D. Bulnes, J.L. López-Bonilla, D.A. Juraev, Klein-Gordon's equation for magnons without non-ideal effect on spatial separation of spin waves, *Stochastic Modelling and Computational Sciences*, **3:1**, 29–37 (2023).

[33] J.D. Bulnes, J.L. López-Bonilla, Dirac's linearization applied to the functional, with matrix aspect, for the time of flight of light, *Maltepe Journal of Mathematics*, **4:2**, 38–43 (2022).

[34] J.D. Bulnes, Solving the heat equation by solving an integro-differential equation, *Global and Stochastic Analysis*, **9:2**, 89–97 (2022).

[35] J.D. Bulnes, F.A. Bonk, On the problem of the theoretical identification of a physical entanglement in spins systems in the NMR quantum computation, *Lat. Am. J. Phys. Educ.*, **9:1**, 1303.1–1303.3 (2015).

[36] J.D. Bulnes, F.A. Bonk, A case of spurious quantum entanglement originated by a mathematical property with a nonphysical parameter, *Latin-American Journal of Physics Education*, **8**, 4306-1–4306-4 (2014).

[37] J.D. Bulnes, L.A. Peche, Entrelazamiento cuántico espúrio con matrizes seudopuras extendidas 4 por 4, *Revista Mexicana de Física*, **57**, 188–192 (2011).

[38] J.D. Bulnes, Propagadores cuánticos calculados de acuerdo con el postulado de Feynman con caminos aproximados por polinomios, *Revista Mexicana de Física*, **55**, 34–43 (2009).

[39] J.D. Bulnes, F.A. Bonk, R.S. Sarthour, E. Azevedo, J.C.C. Freitas, T.J. Bonagamba, I.S. Oliveira, Quantum information processing through nuclear magnetic resonance, *Brazilian Journal of Physics*, **35**, 617–625 (2005).

[40] J.D. Bulnes, I.S. Oliveira, Construction of exact solutions for the Stern Gerlach effect, *Brazilian Journal of Physics*, **31:3**, 488–495 (2001).

[41] I.V Misyurkeyev, *Problems in Mathematical Physics*. McGraw-Hill Book Company, New York, (1966).

[42] V. Eiderman, *An Introduction to Complex Analysis and the Laplace Transform*, Taylor & Francis Group, (2022).

[43] I.N. Sneddon, *Elements of Partial Differential Equations*, McGraw-Hill, New York, (1957).

[44] P.A.M. Dirac, Complex variables in Quantum Mechanics, *Proceedings of the Royal Society A*, **160:900**, 48–59 (1937).

[45] A. Einstein, B. Podolsky, N. Rosen, Can Quantum-Mechanical Description of Physical Reality be considered complete?, *Physical Review*, **47**, 777–780 (1935).

Author information

J.D. Bulnes, 1. Department of Exact and Technological Sciences, Federal University of Amapá, Rod. J. Kubitschek, 68903-419, Macapá, AP, BRAZIL; 2. Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, UZBEKISTAN.
E-mail: `bulnes@unifap.br`

M.A.I. Travassos, Department of Exact and Technological Sciences, Federal University of Amapá, Rod. J. Kubitschek, 68903-419, Macapá, AP, BRAZIL.
E-mail: `angelicaptravass@gmail.com`

D.A. Juraev, 1. Scientific Research Center, Baku Engineering University, Baku AZ0102, AZERBAIJAN; 2. Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, University of Economics and Pedagogy, Karshi, 180100, UZBEKISTAN.
E-mail: `juraevdavron12@gmail.com`

J.L. López-Bonilla, ESIME−Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, MEXICO.
E-mail: `joseluis.lopezbonilla@gmail.com`

Received: 05.11.2024
Accepted: 09.09.2024
Published: 27.12.2024

WARNING: Title too long for running head.
PLEASE supply a shorter form with \headlinetitle