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EXTENSION OF THE GRADIENT EFFECT ON EIGENENERGIES TO THE "ELECTRIC STERN-GERLACH" EFFECT WITH SQUARE OF THE LINEARIZED FIELD

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Abstract: In this paper, we constructed a solution of the Schrödinger equation for the ammonia molecule, modeled as a particle with a permanent electric dipole moment, which can access only two quantum states, as part of a beam of these molecules entering a region where an electrostatic field with a weak gradient act. In this solution, the contribution of the electric field gradient to the eigenenergies of the molecule stands out in the context of linearization of the square of one of the components of the electrostatic field. This result indicates that the analogy established between the spatial separation of this beam of molecules and the spatial separation of silver atoms in the inhomogeneous magnetic field in the Stern-Gerlach effect is not limited only to spatial separation; the gradient effect on eigenenergies, recently identified in the Stern-Gerlach case, also manifests itself in the case of ammonia molecules in an inhomogeneous electrostatic field.

Keywords and phrases: Electric dipole moment, Polarizability matrix, Analogy with the Stern-Gerlach effect.

MSC 2010 Classifications: Primary 34A05; Secondary 15A18.

1 Introduction

In the scientific literature dedicated to discoveries in quantum mechanics, there are some extensions of known problems, such as in [1]-[13], in which we can see the identification of some previously disregarded aspects that escaped analysis[1]. Likewise, in the literature of mathematical physics, there are new findings that show that several mathematical approaches can be adapted to evaluate several complementary aspects in quantum mechanics [14]-[23].

Here we are interested in the so-called "electric Stern-Gerlach" effect, an analog of the Stern-Gerlach effect[2]. We begin by reviewing references [24]-[26], which address the problem of a beam of ammonia molecules (NH_3), initially traveling along the positive X direction (Fig. 1), and then splitting into two secondary beams in the $Y = 0$ plane, within a region where a non-homogeneous electric field acts, whose spatial configuration presents a gradient along the Z coordinate direction.

Regarding references [24]-[26], it is worth noting that these do not contain analytical expressions for the corresponding eigenfunctions; furthermore, the gradient of the electric field does not appear explicitly in their equations, only the square of the electric field, which is considered intense. In [27], on the other hand, considering the ammonia molecule as a system with only two energy levels (of the vibrational type) and the action of a perturbative potential with a specific shape, it was determined that the expressions of the corresponding energy levels depend on the applied electric field, which is why the Stark effect was recognized. It should be mentioned, in the case of [24], that these absences are consistent with a previous comment by the author[3].

[1] An example of this situation is clearly seen in [1].

[2] Which is known and explained via quantum treatment from the interaction between the magnetic moment of the electron spin (in a hydrogen atom of a substance such as vaporized silver) with a magnetic field, generated by a magnet, which produces an intense gradient.

[3] Feynman said: "So I tried to describe the principles of quantum mechanics in a way that didn't require first knowing the

It can be inferred from figure 1 above, and from other similar figures, that the expression "electric Stern-Gerlach" effect, uses the spatial separation of a beam of ammonia molecules into two secondary beams as the only argument to support the analogy with the Stern-Gerlach effect. It is shown here, however, that this is not the case. Based on a result shown in [3],[13], that the gradient of a Stern-Gerlach-type magnetic field explicitly contributes to the eigenenergies of the corresponding Hamiltonian, it is found, in a certain physical context and under certain conditions, that the non-zero gradient of the electrostatic field also contributes to the respective eigenenergies (and not only the gradient, as we will see later), which extends the analogy between both effects.

In this paper, we follow almost completely the general physical framework defined in [24]-[26] for the ammonia molecule, as well as for the corresponding beam. In particular, we model the ammonia molecule as an effective quantum particle[4], without internal structure, with mass equal to the molecule's mass and with the same permanent electric dipole moment. Furthermore, and unlike these references, the electric field gradient appears here explicitly in the equations; in particular, we assume that this gradient is sufficiently weak, so that Δz, in Fig. 1, corresponding to the vertical aperture of the initial beam into two beams, is sufficiently small, with which the linearization of the term proportional to z^2 (z within the interval Δz) is a good approximation. Based on this, we construct the explicit eigenfunctions for the "electric Stern-Gerlach" effect, for the decoupled equations for the state vector components. In addition, we explicitly show that the eigenenergies of the model incorporate the contribution of the electrostatic field's gradient and the density of the electric charges that generate this field; results not found in [24]-[26].

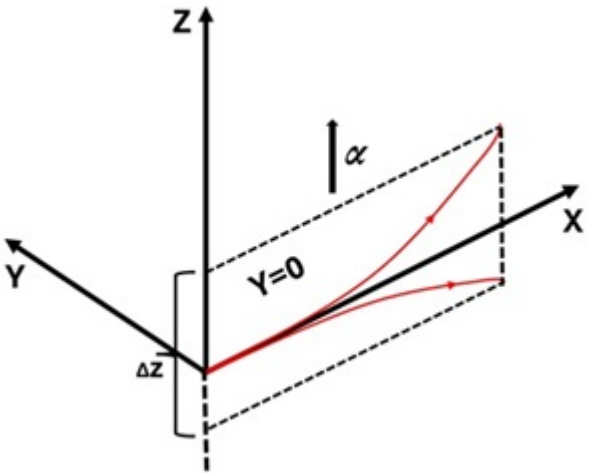

Figure 1. Assuming that the magnitude of the gradient is small enough, the unfolded beams (red) only separate from each other along the vertical. This separation, for better visualization, is shown exaggeratedly in the figure.

Finally, regarding the beams unfolded, contained in the plane $Y = 0$, the following situations are possible: (i) All molecules in a beam are in the same steady state; that is, the molecules in the first beam have a different state from the molecules in the second beam; (ii) In the same beam there are two sets of molecules, each in a steady state. The first case corresponds (see section 3) to the value $A = 0$ and the second to the value $A \neq 0$; in particular, non-zero and small values of A correspond to situations in which there is a majority presence of molecules in the same steady state.

mathematics of partial differential equations".

[4] A point particle, so that in the energy levels there will be no contributions that could come from vibrations, deformations, etc. that can manifest themselves in a non-point structure.

1.1 Representation of the electrostatic field, the electrostatic potential, and parity symmetry

The beam of molecules enters a region (limited by the dashed lines in Fig. 1) where it interacts with an electrostatic field, which is generated by a distribution of electric charge ρ.This field if dependent on the coordinate variables y and z, can be formally expressed as,

$$\vec{\mathcal{E}}(y,z) = -\alpha y\mathbf{j} + \left(\mathcal{E}_0 + (\alpha + \frac{\rho}{\epsilon_0})z\right)\mathbf{k}, \tag{1.1}$$

where $\mathcal{E}_0$ represents the homogeneous component of this field, and α the magnitude of the gradient directed along the Z and Y directions.

The equation of Gauss's law for electrostatics, in the international system of units:

$$\nabla\cdot\vec{\mathcal{E}} = \rho/\epsilon_0\ , \tag{1.2}$$

is verified by (1.1). Furthermore, the field in (1.1) can be derived, according to $\vec{\mathcal{E}} = -\nabla\varphi$, from the scalar potential:

$$\varphi(y,z) = \frac{\alpha}{2}y^2 - \mathcal{E}_0 z - \frac{1}{2}\left(\alpha + \frac{\rho}{\epsilon_0}\right)z^2. \tag{1.3}$$

Independently of the above, let us now see the effect of the parity transformation on the field (1.1). Then, we have,

$$\vec{\mathcal{E}}(y,z) \quad \Rightarrow \quad \vec{\mathcal{E}}'(y',z') \ = \ \vec{\mathcal{E}}(-y,-z) \ = -\ \vec{\mathcal{E}}(y,z) + 2\mathcal{E}_0\hat{k}, \tag{1.4}$$

Thus, it is clear that the field (1.4), for $\mathcal{E}_0 = 0$, satisfies:

$$\vec{\mathcal{E}}(-y,-z) \ = -\ \vec{\mathcal{E}}(y,z), \tag{1.5}$$

which corresponds to a parity transformation for a vector field, so that the field $\vec{\mathcal{E}}$ in (1.1) will be reduced here to the expression,

$$\vec{\mathcal{E}}(y,z) = -\alpha y\hat{j} + \left(\alpha + \frac{\rho}{\epsilon_0}\right)z\hat{k}, \tag{1.6}$$

whose form will be considered as the correct one from now on. On the other hand, let us also see the effect of the parity transformation on the electrostatic potential $\varphi(y,z)$, previously taking $\mathcal{E}_0 = 0$, as already justified; then, from:

$$\varphi(y,z) = \frac{\alpha}{2}y^2 - \frac{1}{2}\left(\alpha + \frac{\rho}{\epsilon_0}\right)z^2\ , \tag{1.7}$$

we have:

$$\varphi(y,z) \quad \Rightarrow \quad \varphi'(y',z') = \varphi(-y,-z) = \frac{\alpha}{2}y^2 - \frac{1}{2}\left(\alpha + \frac{\rho}{\epsilon_0}\right)z^2\ , \tag{1.8}$$

that is, the expression (1.7) is invariant in the face of a parity transformation, thus corresponding to a scalar quantity, as expected for the electrostatic potential:

$$\varphi(-y,-z) = \varphi(y,z). \tag{1.9}$$

2 Mathematical developments

From classical electromagnetism we know that polar molecules, such as ammonia, have a permanent electric dipole moment. This type of molecule undergoes deformation proportional to the acting local field (assuming a linear regime), in the presence of an external electric field, and tends to orient itself parallel to the direction defined by the external field at the location.

On the other hand, in a semi-classical treatment, the components μ_1, μ_2 and μ_3 of $\vec{\mu}$, the induced[5] electric dipole moment in the ammonia molecule (in the presence of an external electric

[5]The only one to be considered here, since this induced electric moment is sufficient to reveal the gradient dependence on the eigenenergies of the Hamiltonian in (2.3).

field $\vec{\mathcal{E}}$, are defined by a polarizability matrix:

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix}. \tag{2.1}$$

In our case, from (1.6), we have that $\mathcal{E}_1 = 0$, $\mathcal{E}_2 = -\alpha y$ and $\mathcal{E}_3 = (\alpha + (\rho/\epsilon_0))z$.

According to expression (10), the components of $\vec{\mu}$ are:

$$\begin{aligned} \mu_1 = \sigma_{12}\mathcal{E}_2 + \sigma_{13}\mathcal{E}_3 \;\Rightarrow\; \mu_1 = -\,\sigma_{12}\Big(\alpha y\Big) + \sigma_{13}\Big(\alpha + (\rho/\epsilon_0)\Big)z, \\ \mu_2 = \sigma_{22}\mathcal{E}_2 + \sigma_{23}\mathcal{E}_3 \;\Rightarrow\; \mu_2 = -\,\sigma_{22}\Big(\alpha y\Big) + \sigma_{23}\Big(\alpha + (\rho/\epsilon_0)\Big)z, \\ \mu_3 = \sigma_{32}\mathcal{E}_2 + \sigma_{33}\mathcal{E}_3 \;\Rightarrow\; \mu_3 = -\,\sigma_{32}\Big(\alpha y\Big) + \sigma_{33}\Big(\alpha + (\rho/\epsilon_0)\Big)z. \end{aligned} \tag{2.2}$$

Furthermore, the quantum Hamiltonian for an "effective particle[6]", with induced electric dipole moment $\vec{\mu}$, in the electrostatic field given in (1.6), is written as:

$$\hat{\mathcal{H}} = \frac{1}{2m}\Big\{\hat{\mathcal{P}}_x^2 + \hat{\mathcal{P}}_y^2 + \hat{\mathcal{P}}_z^2\Big\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \vec{\mu}\cdot\vec{\mathcal{E}} & -A \\ -A & -\vec{\mu}\cdot\vec{\mathcal{E}} \end{pmatrix}, \tag{2.3}$$

where A is the probability amplitude that a particle that is initially in the stationary state φ_1 passes to the stationary state φ_2.

For the rightmost term in (2.3), corresponding to the interaction between the electric dipole moment and the externally applied electric field, we write,

$$\hat{\mathcal{H}}' = \begin{pmatrix} \vec{\mu}\cdot\vec{\mathcal{E}} & -A \\ -A & -\vec{\mu}\cdot\vec{\mathcal{E}} \end{pmatrix}, \tag{2.4}$$

where,

$$\begin{aligned} \vec{\mu}\cdot\vec{\mathcal{E}} \;=\; \Big\{-\sigma_{22}\Big(\alpha y\Big) \;+\; \sigma_{23}\Big(\alpha + (\rho/\epsilon_0)\Big)z\Big\}(\alpha y) \;+ \\ +\Big\{-\sigma_{32}\Big(\alpha y\Big) + \sigma_{33}\Big(\alpha + (\rho/\epsilon_0)\Big)z\Big\}(\alpha + \frac{\rho}{\epsilon_0})z. \end{aligned} \tag{2.5}$$

In the case drawn in Fig.1, and as considered in [14, 15, 16], there is no gradient along the Y direction, and the separation of the initial beam into two beams only occurs in the plane $Y = 0$ (from which it follows, according to (1.6), that $\mathcal{E}_2 = 0$); in this case, expression (2.4) is simplified to the following,

$$\mathcal{H}'' = \begin{pmatrix} \sigma_{33}\Big(\alpha + (\rho/\epsilon_0)\Big)^2 z^2 & -A \\ -A & -\sigma_{33}\Big(\alpha + (\rho/\epsilon_0)\Big)^2 z^2 \end{pmatrix}. \tag{2.6}$$

For what follows, and this is a key procedure in this development, we proceed to linearize the term: $\big(\alpha + (\rho/\epsilon_0)\big)^2 z^2$, which corresponds (except for the coefficient $-\sigma_{33}$) to one of the elements on the diagonal of the matrix $\mathcal{H}''$, in (2.6), within a "suitable" interval. Then we do:

$$\Big(\alpha + (\rho/\epsilon_0)\Big)^2 z^2 \;\rightarrow\; \Big((\alpha + (\rho/\epsilon_0))z\Big)^2 \;\rightarrow\; (\omega)^2 \;\rightarrow\; a\omega + \tilde{b} \;=\; \Big(\alpha + (\rho/\epsilon_0)\Big)\Big(az + b\Big).$$

where we have performed the linearization in the third step, from left to right, in the schematic sequence above, keeping the factor $\big(\alpha + (\rho/\epsilon_0)\big)$ explicit.

[6]Corresponding to an ammonia molecule, as discussed previously.

In this situation, where we use the linear approximation above, we have a simpler expression for the electrical interaction matrix (i.e., the interaction between the induced electric dipole moment and the external electrostatic field):

$$\mathcal{H}''_{linear} = \begin{pmatrix} \sigma_{33}\left(\alpha + \frac{\rho}{\epsilon_0}\right)(az+b) & -A \\ -A & -\sigma_{33}\left(\alpha + \frac{\rho}{\epsilon_0}\right)(az+b) \end{pmatrix}. \tag{2.7}$$

Thus, for the approach and assumptions considered, the stationary equation,

$$\hat{\mathcal{H}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \left\{ \frac{1}{2m}(\hat{\mathcal{P}}_x^2 + \hat{\mathcal{P}}_z^2)I + \mathcal{H}''_{linear} \right\} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = E \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2.8}$$

where I is the identity matrix, can be written explicitly as a system of two coupled[7] equations for the components φ_1 and φ_2 of the (non-relativistic) spinor,

$$\left\{ \frac{1}{2m}(\hat{\mathcal{P}}_x^2 + \hat{\mathcal{P}}_z^2) + \sigma_{33}\left(\alpha + (\frac{\rho}{\epsilon_0})\right)(az+b) \right\} \varphi_1 - A\varphi_2 = E_1\varphi_1 \tag{2.9}$$

$$\left\{ \frac{1}{2m}(\hat{\mathcal{P}}_x^2 + \hat{\mathcal{P}}_z^2) - \sigma_{33}\left(\alpha + (\frac{\rho}{\epsilon_0})\right)(az+b) \right\} \varphi_2 - A\varphi_1 = E_2\varphi_2 \tag{2.10}$$

or, after grouping like terms, we have:

$$\left\{ \frac{1}{2m}(\hat{\mathcal{P}}_x^2 + \hat{\mathcal{P}}_z^2) + a\ \sigma_{33}\left(\alpha + (\frac{\rho}{\epsilon_0})\right)z \right\} \varphi_1 - A\varphi_2 = \tilde{E}_1\varphi_1 \tag{2.11}$$

$$\left\{ \frac{1}{2m}(\hat{\mathcal{P}}_x^2 + \hat{\mathcal{P}}_z^2) - a\ \sigma_{33}\left(\alpha + (\frac{\rho}{\epsilon_0})\right)z \right\} \varphi_2 - A\varphi_1 = \tilde{E}_2\varphi_2. \tag{2.12}$$

where the eigenenergies are redefined (which now include a dependence on the gradient), as follows,

$$\tilde{E}_1 \equiv E_1 - \sigma_{33}\left(\alpha + (\frac{\rho}{\epsilon_0})\right)b, \tag{2.13}$$

$$\tilde{E}_2 \equiv E_2 + \sigma_{33}\left(\alpha + (\frac{\rho}{\epsilon_0})\right)b. \tag{2.14}$$

Equations (20) and (21) can also be written compactly as,

$$\hat{\mathcal{O}}_1\varphi_1 - A\varphi_2 = \tilde{E}_1\varphi_1 \tag{2.15}$$

$$\hat{\mathcal{O}}_2\varphi_2 - A\varphi_1 = \tilde{E}_2\varphi_2 \tag{2.16}$$

where $\hat{\mathcal{O}}$ is the corresponding differential operator. In the next section we will construct the solutions to equations (20) and (21) assuming the particular case when there is no coupling ($A = 0$) between them.

3 Solution for the decoupled case of the equations for the two components of the state vector

Before we begin to solve equations (2.11) and (2.12), for $A = 0$, it should be noted that our purpose is to determine whether or not the gradient of the electrostatic field contributes to the eigenenergies of the physical system considered.

The independent equations that we have to consider are:

$$\left\{ \frac{1}{2m}(\hat{\mathcal{P}}_x^2 + \hat{\mathcal{P}}_z^2) \ + \ a\ \sigma_{33}(\alpha + \frac{\rho}{\epsilon_0})z \right\}\varphi_1 \ = \ \tilde{E}_1\varphi_1, \tag{3.1}$$

[7] In [10] what was initially considered a system of two coupled Schrödinger equations was considered, showing that they admitted exact decoupling, but for two functions distinct from the initial ones.

$$\left\{ \frac{1}{2m}(\hat{\mathcal{P}}_x^2+\hat{\mathcal{P}}_z^2) \ - \ a\ \sigma_{33}\big(\alpha+\frac{\rho}{\epsilon_0}\big)z \right\}\varphi_2 \ = \ \tilde{E}_2\varphi_2. \tag{3.2}$$

We begin with equation (3.1). Assuming that a solution has the form corresponding to the separation of its variables, we write:

$$\varphi_1(x,z)=X(x)\ Z(z), \tag{3.3}$$

Next, since the incident beam moves along the X direction and the unfolded beams move away from each other symmetrically with respect to the same direction, we will impose, given the presence of the leftmost term in (3.1), that: $X(x)=e^{(i/\hbar)p_x x}$, where p_x is an eigenvalue of $\hat{\mathcal{P}}_x$. In this case, we obtain the following equation for the function Z:

$$Z''(z) \ - \ \left\{ \frac{1}{\hbar^2}\left(2m\ a\ \sigma_{33}\Big(\alpha+(\rho/\epsilon_0)\Big)\right)z-\frac{2m}{\hbar^2}\Big(\tilde{E}_1-\frac{p_x^2}{2m}\Big)\right\}Z(z) \ = \ 0 \tag{3.4}$$

where, for brevity, we write:

$$\lambda\equiv\frac{1}{\hbar^2}\left(2m\ a\ \sigma_{33}\Big(\alpha+(\rho/\epsilon_0)\Big)\right), \quad \lambda_1\equiv\frac{2m}{\hbar^2}\left(\tilde{E}_1-\frac{p_x^2}{2m}\right). \tag{3.5}$$

Note that the expression in (29) is an equation not so different from the Airy equation. We can arrive at the same Airy equation by changing the variable: $\zeta=\lambda^{1/3}z-\lambda_1\lambda^{-2/3}$; that is, we go from (3.4) to the equation:

$$\mathcal{Z}''(\zeta) \ - \ \zeta\mathcal{Z}(\zeta) \ = \ 0, \tag{3.6}$$

the well-known Airy equation; that is, $\mathcal{Z}\equiv Ai$. Then we have, from (3.3):

$$\varphi_1(x,z)=e^{(i/\hbar)p_x x}\ Ai\Big(\lambda^{1/3}z+\xi_0\Big). \tag{3.7}$$

where ξ_0, representing the coordinate of the point of an absolute maximum of the Airy function (Ai), is incorporated in (3.7) to move this maximum to the origin of coordinates (the same procedure will be done, but symmetrically, for the function φ_2), so that the absolute maximum of φ_1 and φ_2 coincide since their separation is only due to their interaction with the electrostatic field. Note, according to what has already been seen, that one would expect the function Ai in (3.7) to have as an argument the expression $(\lambda^{1/3}z-\lambda_1\lambda^{-2/3})$, but this is not the case; the correct expression is $(\lambda^{1/3}z+\xi_0)$.

On the other hand, it is known that the Airy function presents, on one of its sides, decreasing oscillations that do not vanish at infinity and, on its opposite side, a continuously decreasing behavior that begins at its absolute maximum. Also, from (3.7), we have that the function: $|\varphi_1|^2=|Ai|^2$ is not integrable. Therefore, to represent a physically acceptable wave function, we should eliminate the part of the function Ai that produces this inconvenience. Using the same resource used in [13], that is: we appropriately multiply the function in (3.7) by the Heaviside function (Θ).

$$\varphi_1(x,z)=e^{(i/\hbar)p_x x}\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\ \Theta\Big(z-\frac{\lambda_0}{\lambda^{1/3}}+\xi_0\Big), \tag{3.8}$$

which cuts the Airy function at its first zero, λ_0.

With regard to determining the eigenenergy $\tilde{E}_1$, or E_1, it is sufficient to consider (3.7). So, after calculating the spatial derivatives of the function in (3.7), we can write:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial z^2}\right)+a_l\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)z\ \varphi_1=$$

$$=\ \frac{p_x^2}{2m}\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}-\Big(\frac{\hbar^2\lambda^{2/3}}{2m}\Big)Ai''\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}+$$

$$+\ a_l\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)z\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}. \tag{3.9}$$

In (3.9), we will add and subtract the quantity:

$$\xi_0\Big\{a^2\ \sigma_{33}^2\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)^2\hbar^2/2m\Big\}^{1/3}Ai\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}, \tag{3.10}$$

Then we have:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi_1}{\partial x^2}+\ \frac{\partial^2\varphi_1}{\partial z^2}\right)+a\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)z\ \varphi_1=$$

$$=\ \left(\frac{p_x^2}{2m}-\xi_0\Big\{a^2\ \sigma_{33}^2\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)^2\hbar^2\Big/2m\Big\}^{1/3}\right)\ Ai\Big(\lambda^{1/3}+\xi_0\Big)\ e^{(i/\hbar)p_x x}+$$

$$-\Big(\frac{\hbar^2\lambda^{2/3}}{2m}\Big)Ai''\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}+$$

$$+\left(a_l\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)z\ +\ \xi_0\Big\{a^2\ \sigma_{33}^2\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)^2\hbar^2\Big/2m\Big\}^{1/3}\right)\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}. \tag{3.11}$$

or, also:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi_1}{\partial x^2}+\ \frac{\partial^2\varphi_1}{\partial z^2}\right)+a\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)z\ \varphi_1=$$

$$=\ \left(\frac{p_x^2}{2m}-\xi_0\Big\{a^2\ \sigma_{33}^2\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)^2\hbar^2\Big/2m\Big\}^{1/3}\right)\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}+$$

$$-\Big(\frac{\hbar^2\lambda^{2/3}}{2m}\Big)\Big\{Ai''\Big(\lambda^{1/3}z+\xi_0\Big)-\kappa\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\Big\} \tag{3.12}$$

where,

$$\kappa=\left(\left\{\frac{2m\ a\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)}{\hbar^2\lambda^{2/3}}\right\}z+\xi_0\Big(\frac{2m}{\hbar^2\lambda^{2/3}}\Big)\left\{\frac{a^2\ \sigma_{33}^2\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)^2\hbar^2}{2m}\right\}^{1/3}\right). \tag{3.13}$$

Considering (3.5), we find that κ is rewritten as:

$$\kappa=\lambda^{1/3}z+\xi_0. \tag{3.14}$$

Thus, equation (37) can be written as follows:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi_1}{\partial x^2}+\ \frac{\partial^2\varphi_1}{\partial z^2}\right)+a\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)z\ \varphi_1=$$

$$=\ \left(\frac{p_x^2}{2m}-\xi_0\Big\{a^2\ \sigma_{33}^2\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)^2\hbar^2\Big/2m\Big\}^{1/3}\right)\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\ e^{(i/\hbar)p_x x}+$$

$$-\Big(\frac{\hbar^2\lambda^{2/3}}{2m}\Big)\Big\{Ai''\Big(\lambda^{1/3}z+\xi_0\Big)-\Big(\lambda^{1/3}z+\xi_0\Big)\ Ai\Big(\lambda^{1/3}z+\xi_0\Big)\Big\}. \tag{3.15}$$

But, the second term on the right-hand side in equation (3.15) equals zero, since $Ai\Big(\lambda^{1/3}z+\xi_0\Big)$ satisfies the Airy equation. Therefore, we have the simplified expression:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi_1}{\partial x^2}+\ \frac{\partial^2\varphi_1}{\partial z^2}\right)+a\ \sigma_{33}\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)z\ \varphi_1= \tag{3.16}$$

$$=\ \left(\frac{p_x^2}{2m}-\xi_0\Big\{a^2\ \sigma_{33}^2\Big(\alpha+\frac{\rho}{\epsilon_0}\Big)^2\hbar^2\Big/2m\Big\}^{1/3}\right)\ \varphi_1.$$

Comparing (41) with (26), we have that the eigenenergy associated with the function φ_1 is given by:

$$\tilde{E}_1 = \frac{p_x^2}{2m} - \xi_0 \left\{ \frac{a^2\, \sigma_{33}^2 \Big(\alpha + (\rho/\epsilon_0)\Big)^2 \hbar^2}{2m} \right\}^{1/3}. \tag{3.17}$$

Replacing the expression $\tilde{E}_1 \equiv E_1 - \sigma_{33}\Big(\alpha + (\frac{\rho}{\epsilon_0})\Big)b$, found in (2.13), in expression (3.17), we obtain the following formal expression for the eigenenergy E_1:

$$E_1 = \frac{p_x^2}{2m} + \sigma_{33}\Big(\alpha + (\frac{\rho}{\epsilon_0})\Big)b - \xi_0 \left\{ \frac{a^2\, \sigma_{33}^2 \Big(\alpha + (\rho/\epsilon_0)\Big)^2 \hbar^2}{2m} \right\}^{1/3}. \tag{3.18}$$

Following a similar procedure, we obtain for φ_2:

$$\varphi_2(x,z) = e^{(i/\hbar)p_x x}\, Ai\Big(-\lambda^{1/3} z + \xi_0\Big)\, \Theta\Big(-z - \frac{\lambda_0}{\lambda^{1/3}} + \xi_0\Big), \tag{3.19}$$

With eigenenergy:

$$E_2 = \frac{p_x^2}{2m} - \sigma_{33}\Big(\alpha + (\frac{\rho}{\epsilon_0})\Big)b - \xi_0 \left\{ \frac{a^2\, \sigma_{33}^2 \Big(\alpha + (\rho/\epsilon_0)\Big)^2 \hbar^2}{2m} \right\}^{1/3}, \tag{3.20}$$

where it is clear that the eigenenergies depend on the magnitude of the gradient α and the electric charge density ρ. These results are valid only within the context of linearization that we have assumed.

4 Discussion

In the context that we have considered, we note, on the one hand, that in (3.18) and (3.20) there are two types of contributions of the electrostatic field gradient to the eigenenergies; on the one hand, the term:

$$\pm\, \sigma_{33}\Big(\alpha + (\frac{\rho}{\epsilon_0})\Big)b,$$

produces the energetic separation between the molecules, in relation to the energies of the translational movement, while the term,

$$-\, \xi_0 \left\{ \frac{a^2\, \sigma_{33}^2 \Big(\alpha + (\rho/\epsilon_0)\Big)^2 \hbar^2}{2m} \right\}^{1/3},$$

produces the same displacement of the energetic levels unfolded by the linear term in: $\alpha + (\rho/\epsilon_0)$.

In the case with a very weak gradient, we have that the spatial separation of the initial beam of molecules into two beams, in the plane $Y = 0$, in the region where the electric field acts, would be small.

On the other hand, we have from (3.18) and (3.20), that for the values of the geometric parameters $a = 0$ and $b = 0$, only the kinetic energy term remains in the eigenenergies, which means that these values would characterize, in an operational manner, and after the linearization of the quadratic term, the absence of an electric field acting on the beam of molecules; and then we would have to:

$$E_1 = E_2 = \frac{p_x^2}{2m}.$$

Also note an interesting fact. In the magnetic case, the density of magnetic charges can only have a null value[8], so that taking the value $\alpha = 0$ is enough to cancel the gradient effect

[8] Since magnetic monopoles do not exist as elementary particles; however, magnetic monopoles resulting from a collective effect have been experimentally observed in the structure called "spin ice" [28].

on the eigenenergies in the Stern-Gerlach effect [9]. On the other hand, in the "electric Stern-Gerlach" case, the density of electric charges is non-zero; thus, $\alpha = 0$, alone, could not cancel the additional effect of displacement in the energy levels given in (3.18) and (3.20), unless, together with a null value attributed to the density of electric charges $\rho = 0$ (which would then not generate an electrostatic field). In other words, in the case of the "electric Stern-Gerlach" effect, the gradient effect on eigenenergies becomes a combined effect of the gradient and the density of electric charges, a type of result that the authors claim to be unaware of in the general literature on quantum mechanics.

5 Conclusion

This article considers a beam of ammonia molecules within a region where an inhomogeneous electric field acts. These molecules were modeled as quantum particles that can access only two stationary states; without internal structure, however, with a permanent electric dipole moment. Assuming a weak gradient regime and via a linearization process of a quadratic term, we obtain explicit analytical expressions, (3.8) and (3.19), for the eigenfunctions of the reduced Hamiltonian (2.7), as well as for the corresponding eigenenergies, (3.18) and (3.20), which present explicit dependence on the gradient of the electrostatic field and the density of electric charges that generate this field, results that do not appear in the references [14, 15, 16].

The results found for the eigenenergies, in particular, show that the analogy between the spatial separation observed in the Stern-Gerlach effect and the spatial separation of a beam of ammonia molecules, into two secondary beams due to their interaction with this electrostatic field, is broader than considered, since an analogy can also be established between the corresponding eigenenergies, in the sense of the manifestation of the gradient effect in both cases.

The spatial separation of the incident beam into two secondary beams, in the "electric Stern-Gerlach" effect, results from the molecules presenting a permanent electric dipole moment. Here, we were particularly interested in the energetic part of this effect, where it was sufficient to consider the induced electric dipole moment.

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